



Flag manifold σ -models: The $\frac{1}{N}$ -expansion and the anomaly two-form

Dmitri Bykov ^{a,b,*}

^a *Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 Munich, Germany*

^b *Steklov Mathematical Institute of Russ. Acad. Sci., Gubkina str. 8, 119991 Moscow, Russia*

Received 18 January 2019; accepted 7 February 2019

Available online 13 February 2019

Editor: Hubert Saleur

Abstract

We construct a gauged linear sigma-model representation and develop a $\frac{1}{N}$ -expansion for flag manifold σ -models previously proposed by the author. Classically there exists a zero-curvature representation for the equations of motion of these models, which leads in particular to the existence of a conserved non-local charge. We show that at the quantum level this charge is no longer conserved and calculate explicitly the anomaly in its conservation law.

© 2019 The Author. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

The subject of integrability in two-dimensional field theory has a long history. After the initial success with the Korteweg–de Vries equation (the main developments being summarized in [1]), similar technology, based on the zero-curvature representation, was applied to relativistic σ -models with symmetric target spaces [2,3]. In classical theory the mathematical groundwork was laid in [4,5]. In quantum theory the initial advances in the sine-Gordon theory [6,7] were succeeded by the calculation of the S-matrix of the S^{N-1} sigma-model [8]. For an introduction to the subject of quantum-integrable sigma-models with symmetric target spaces see the lectures [9].

Around the same time the $\frac{1}{N}$ -expansion was formulated for the S^{N-1} -model in [8] and for the CP^{N-1} -model in [10]. The latter model was found to be non-integrable due to anomalies in the

* Correspondence to: Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 Munich, Germany.
E-mail addresses: bykov@mpp.mpg.de, bykov@mi-ras.ru.

higher local conservation laws [11], whose potential appearance had been discussed in [12]. The Hamiltonian structure of the flat connections of the model was discussed in [13,14], where in particular involutivity of the local charges was verified. Another approach to the question of integrability was developed in [15] and was based on the non-local conserved charges, which follow from the zero-curvature representation. These charges, if unobstructed by anomalies, generate non-abelian algebraic structures such as the Yangian, that underlie the integrable structure of the theory, as was elaborated in [16]. In theories like the $\mathbb{C}\mathbb{P}^{N-1}$ -model, however, the conservation equation of the first non-local charge is spoiled by an anomaly [17]. This anomaly is also present in various sigma-models with symmetric target spaces [18] but is cancelled in certain fermionic generalizations (see [19] and references therein), for example in the supersymmetric $\mathbb{C}\mathbb{P}^{N-1}$ -model [20–22].

What fell outside of the scope of these classical papers is the case of homogeneous but *non-symmetric* target spaces. These models are in general believed to be not integrable even at the classical level. The technical reason is that the Noether current of such models is in general not flat, which makes it impossible to construct a zero-curvature representation by a standard procedure of [2]. To start with, symmetric spaces, which may also be called \mathbb{Z}_2 -symmetric spaces, allow for a direct generalization, the so-called \mathbb{Z}_m -symmetric spaces. These arise in various contexts in differential geometry, for example in the theory of nearly Kähler and twistor spaces, see Section 3 below for more details. In certain cases sigma-models with such target spaces might be integrable [23,24]. The discussion of \mathbb{Z}_m -symmetric spaces in the context of integrable sigma-models can be found in the mathematical literature in the book [25].

Other important advances in the direction of integrable non-symmetric space sigma-models are related to integrable deformations of the principal chiral model, in particular in the work [26] and in [27]. The deformation of the latter paper is now known as the Yang–Baxter or η -deformation. The algebraic structure behind such deformations was understood in [28], and the Hamiltonian structure of the corresponding models was extensively analyzed in [29,30]. Another approach (proposed by the author) is based on integrable complex structures on the target spaces [31–35]. Some of the most interesting complex homogeneous spaces are the flag manifolds. These manifolds are coincidentally also \mathbb{Z}_m -graded spaces, and the corresponding sigma-models may be understood, in certain cases, as limits of the η -deformed models [36]. Generally speaking, flag manifolds are ubiquitous objects in representation theory, due to the Borel–Weil–Bott theorem (we dedicate Appendix A to the discussion of this point), and also feature in certain supersymmetric constructions, such as harmonic superspace [37] or supersymmetric quantum mechanics [38].

Although this is not directly related to the main line of the present paper, it should be noted that in recent years there is a strong interest in systems with $SU(N)$ -symmetry ($N > 2$) from the point of view of condensed matter physics. This is related to the fact that such systems have now been experimentally realized in systems of cold atoms (see [39], for example). Some of the most well studied systems in statistical physics are the one-dimensional spin chains, which in appropriate limits can be described by flag manifold sigma-models, as shown in [40–42]. The phase structure of such models, in particular the IR effective field theory for various values of parameters, has been studied in [43,44]. It should be noted that these models in general are likely to be non-integrable and differ from the sigma-models studied in the present paper by a choice of B -field. One interesting common feature, though, is the role played by the \mathbb{Z}_m -symmetry. In flag manifold sigma-models m is the number of steps in the flag. From the point of view of spin chains, m is the length of the elementary cell, and the \mathbb{Z}_m -symmetry arises from shifting the elementary cell by one site. In the context of integrable models discussed in this paper, the

group \mathbb{Z}_m acts on the complex structure defining the B -field of the model, see the proposition in Section 4.

The paper is organized as follows. We start in Section 1 by reminding facts about the geometry of flag manifolds, pertinent to the discussion of the sigma-models in question. In particular, we describe the general procedure of constructing invariant symplectic forms and complex structures in Sections 1.2 and 1.3 respectively. Cohomology is discussed in Section 1.4. We briefly introduce the nonlinear action of the relevant sigma-models in Section 2, for more details the reader is referred to our earlier papers, for example [31,32]. Subsequently in section 3 we prove the relation between our models and the models with \mathbb{Z}_m -graded target spaces introduced in [24]. It turns out that the latter models are essentially independent of the grading, which enters only through topological terms. In Section 4 we construct the gauged linear sigma-model (GLSM) representation for our models (here the \mathbb{Z}_m -symmetry plays an important role). For the case of Kähler metrics on flag manifolds the GLSM-representations were constructed in [45,46], however the peculiarity of our case is that the chosen metric is not Kähler. In Section 5 the Feynman rules for the $\frac{1}{N}$ -expansion of the simplest model with target space $\frac{U(N)}{U(1) \times U(1) \times U(N-2)}$ are derived. We pass on to the discussion of the Wilson loop of the one-parametric family of flat connections in Section 6 and propose a novel prescription for the regularized non-local charge in Subsection 6.1. Then we study the limit when the regularization parameter ϵ vanishes. This limit depends on the operator product expansions of the Noether currents, which are calculated in Section 6.2. In particular, the OPE of two holomorphic components $[K_z(z + \epsilon), K_z(z)]$ is necessary to show that the limit exists, and the OPE of the holomorphic/anti-holomorphic components $[K_z(z + \epsilon), K_{\bar{z}}(z)]$ characterizes the anomaly two-form of the non-local charge. The latter OPE is what replaces the zero-curvature condition for the current K in the quantum theory. In the Appendix we provide some details of the calculations (in which case the relevant appendices are referred to in the body of the paper), as well as discuss certain auxiliary, but nevertheless interesting aspects of the theory of flag manifolds.

1. The geometry of flag manifolds

1.1. The flag manifold as a complex manifold

The flag manifold in \mathbb{C}^N may be defined as the manifold of linear complex subspaces, embedded to each other:

$$\mathcal{F}(d_1, \dots, d_m) = \{0 \subset V_1 \subset \dots \subset V_{m-1} \subset V_m = \mathbb{C}^N\}, \tag{1}$$

where $\dim V_i = d_i$. The group $GL(N, \mathbb{C})$ acts transitively on this manifold, and the stabilizer of any given point (i.e. of a given sequence of embedded linear spaces) is a maximal parabolic subgroup (‘a staircase’), consisting of matrices, depicted in Fig. 1.

Therefore we can view the flag manifold as a homogeneous space

$$\mathcal{F}(d_1, \dots, d_m) = GL(N, \mathbb{C})/\mathcal{H}. \tag{2}$$

It follows that the flag manifold is a complex manifold of complex dimension

$$\dim_{\mathbb{C}} \mathcal{F}(d_1, \dots, d_m) = N^2 - \sum_{i=1}^m d_i (d_i - d_{i-1}) \tag{3}$$

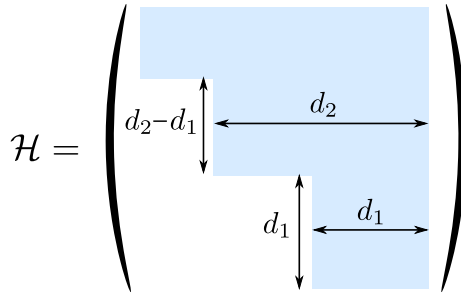


Fig. 1. The maximal parabolic subgroup, stabilizing a flag.

Let us denote $\dim V_i / V_{i-1} = n_i$. Then one can also present the flag manifold as a quotient space of the unitary group:

$$\mathcal{F}_{n_1, \dots, n_m} = \frac{U(N)}{U(n_1) \times \dots \times U(n_m)}, \quad \sum_{i=1}^m n_i = N. \tag{4}$$

This parametrization is naturally related to a choice of Euclidian metric in \mathbb{C}^N , which allows performing orthogonal decompositions of the form $V_i \simeq V_{i-1} \oplus V_i / V_{i-1}$. Let us compute the real dimension and make sure it is twice greater than the complex dimension computed earlier:

$$\dim_{\mathbb{R}} \mathcal{F}_{n_1, \dots, n_m} = N^2 - \sum_{i=1}^m n_i^2 = 2 \times \dim_{\mathbb{C}} \mathcal{F}_{d_1, \dots, d_m}. \tag{5}$$

Note that sometimes we will denote by \mathcal{F}_N the complete flag manifold, i.e. the manifold (4), where all $n_i = 1$.

Example. The projective space $\mathbb{C}P^{N-1}$ may be viewed as the quotient space $(u_1, \dots, u_N) \sim \lambda(u_1, \dots, u_N)$ of $\mathbb{C}^N / \{0\}$. By analogy the Grassmannian $G(k, N)$ is the quotient of the space of $k \times N$ -matrices of rank k w.r.t. the group $GL(k, \mathbb{C})$.

We note that the ‘holomorphic’ definition (2) already implies a choice of complex structure on the flag manifold. The definition (4), in contrast, treats the flag manifold as a real manifold (a complex structure may be defined on it separately – we will do it below).

1.2. Symplectic structures

Let us show that $\mathcal{F}_{n_1, \dots, n_m}$ is a symplectic manifold, i.e. there exists on it a closed 2-form ω , $d\omega = 0$, such that $\omega^{\dim_{\mathbb{C}} \mathcal{F}} \neq 0$ (i.e. it is non-degenerate, $\det \omega \neq 0$). In fact, we will be only considering homogeneous (i.e. $U(N)$ -invariant) symplectic forms.

More generally, let us consider the homogeneous space G/H and the corresponding Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. A homogeneous space is called reductive if the following relations are fulfilled:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}. \tag{6}$$

The second requirement is automatically satisfied, if there is an ad-invariant metric on \mathfrak{g} . Indeed, let there be elements $h_0, h_1 \in \mathfrak{h}$ and $a, b \in \mathfrak{m}$, such that $[h_0, a] = b + h_1$. Let us compute the

scalar product with an arbitrary element $\gamma \in \mathfrak{h}$, then we will get $\langle \gamma, [h_0, a] \rangle = \langle \gamma, h_1 \rangle$. The l.h.s. is zero (which can be verified by swapping the commutator), therefore $\langle \gamma, h_1 \rangle = 0$ for all γ . It follows from the non-degeneracy of the metric that $h_1 = 0$.

There is the following general statement:

G -invariant tensors on a reductive homogeneous space G/H are in correspondence with H -invariant tensors on \mathfrak{m} (the latter is simply a linear space $\mathbb{R}^{\dim \mathfrak{m}}$).

In what follows we will be mainly using the unitary representation (4) and the corresponding Lie algebra decomposition

$$\mathfrak{g} := \mathfrak{u}_N = \bigoplus_{i=1}^m \mathfrak{u}_{n_i} \oplus \mathfrak{m}, \tag{7}$$

where \mathfrak{m} is the orthogonal complement to \mathfrak{h} w.r.t. an ad-invariant metric on $\mathfrak{u}_N = \mathfrak{u}_1 \oplus \mathfrak{su}_N$. Since $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, the subgroup H is represented in the space \mathfrak{m} , and this representation may be decomposed into irreducibles:

$$\mathfrak{m} \otimes \mathbb{C} = \bigoplus_{i \neq j} V_{ij}, \quad \text{where} \quad V_{ij} = \mathbb{C}^{n_i n_j} \tag{8}$$

Moreover, $V_{ji} = \bar{V}_{ij}$. V_{ij} is the vector space of the bi-fundamental representation of the group $U(n_i) \times U(n_j)$.

We introduce the Maurer–Cartan current $j = -g^{-1} dg$, $g \in U(N)$, and decompose it as follows:

$$j = [j]_{\mathfrak{h}} + \sum_{i \neq k} j_{ik}, \quad j_{ik} \in V_{ik}. \tag{9}$$

According to the general statement mentioned above, the most general invariant two-form is

$$\Omega = \sum_{i < j} a_{ij} \text{Tr}(j_{ij} \wedge j_{ji}). \tag{10}$$

To check, in which case it is closed, we will take advantage of the flatness of the Maurer–Cartan current, $dj - j \wedge j = 0$. It follows that

$$\mathcal{D}j_{ij} = \sum_{m \neq (i,j)} j_{im} \wedge j_{mj} \tag{11}$$

where \mathcal{D} is the H -covariant derivative, defined as follows: $\mathcal{D}j_{ij} := dj_{ij} - \{j_{\mathfrak{h}}, j_{ij}\}$. From the condition that Ω is closed it follows that

$$a_{ij} + a_{jk} + a_{ki} = 0 \quad \text{for all pairwise different} \quad (i, j, k). \tag{12}$$

The general solution to this equation is

$$a_{ij} = z_i - z_j. \tag{13}$$

Therefore we have a family of homogeneous symplectic forms with $m - 1$ real parameters. These forms may be compactly written as follows:

$$\Omega = \text{Tr}(z j \wedge j), \quad \text{where} \quad z = \text{Diag}(z_1 \mathbb{1}_{n_1}, \dots, z_m \mathbb{1}_{n_m}). \tag{14}$$

The element z may be normalized to be traceless: $\text{Tr}(z) = 0$. The stabilizer H may now be thought of as the stabilizer of the matrix $z \in \mathfrak{u}_N$, and the flag manifold itself – as an adjoint orbit:

$$\mathcal{F}_{d_1, \dots, d_m} = \{g z g^{-1}, \quad g \in SU(N)\}. \tag{15}$$

The above formula gives an embedding of the flag manifold into the Lie algebra \mathfrak{su}_N . Moreover, this embedding may be identified with the image of the moment map

$$\mu = g z g^{-1}. \tag{16}$$

The action of the group $U(N)$ on the flag manifold, $x \rightarrow \tilde{x}$, is as follows:

$$g_0 \cdot g(x) = g(\tilde{x}) \cdot h_0, \quad g_0 \in U(N), \quad h_0 \in U(n_1) \times \cdots \times U(n_m). \tag{17}$$

The moment map (16) transforms correctly under the action of the group: $\mu(g_0 \cdot g) = g_0 \mu(g) g_0^{-1}$. To check that (16) indeed generates Hamiltonian functions for the action of the group G on the flag manifold, let us assume that v_a is a vector field on \mathcal{F} , corresponding to the Lie algebra element T_a . Then the following holds:

$$j(v_a) \equiv i_{v_a} j = -g^{-1} \nabla_{v_a} g = -g^{-1} \frac{d}{dt} \left(e^{T_a t} g \right) \Big|_{t=0} = -g^{-1} T_a g \tag{18}$$

Using this equality, we can check that (16) indeed gives Hamiltonian functions for the action of the group:

$$d(\text{tr}(\mu T_a)) = d(\text{tr}(g z g^{-1} T_a)) = \text{tr}(z [j, g^{-1} T_a g]) = i_{v_a} \text{tr}(z j \wedge j) = i_{v_a} \Omega \tag{19}$$

1.3. Complex structures

A very detailed treatment of complex structures on homogeneous spaces was given as early as in the classic work [47], so here we mostly present an adaptation of some of these statements to our needs. To start with, on the manifold $\frac{U(N)}{U(1)^N}$ of complete flags in \mathbb{C}^N there are $2^{\frac{N(N-1)}{2}}$ invariant almost complex structures, $N! \leq 2^{\frac{N(N-1)}{2}}$ of them being integrable (we note that for large N , according to Stirling’s formula, $e^N \log(N) < e^{\frac{\log(2) N^2}{2}}$).

Since the complex structure \mathcal{J} is a certain invariant tensor on G/H , according to the logic mentioned in the previous section, one needs to define an H -invariant action of the operator \mathcal{J} on the linear space \mathfrak{m} . Since $\mathcal{J}^2 = -1$, the complex structure may be diagonalized over the complex numbers, in which case we need to define a decomposition

$$\mathfrak{m} \otimes \mathbb{C} = \mathfrak{m}_+ \oplus \mathfrak{m}_-, \tag{20}$$

$$\text{where } [\mathfrak{h}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm. \tag{21}$$

Here \mathfrak{m}_\pm play the role of holomorphic tangent spaces to G/H , i.e. $\mathcal{J} \circ a = \pm i a$ for $a \in \mathfrak{m}_\pm$. In (8) we decomposed $\mathfrak{m} \otimes \mathbb{C}$ into irreducible components, therefore, in order to define an almost complex structure on \mathcal{F} , one can define the action of \mathcal{J} as follows:

$$\mathcal{J} \circ V_{pq} = \pm i V_{pq} \quad \text{for } 1 \leq p < q \leq N. \tag{22}$$

As a result, one has exactly $2^{\frac{N(N-1)}{2}}$ possibilities. There are several equivalent definitions of integrability of a complex structure:

- Vanishing of the Nijenhuis tensor:

$$[\mathcal{J} \circ X, \mathcal{J} \circ Y] - \mathcal{J} \circ ([\mathcal{J} \circ X, Y] + [X, \mathcal{J} \circ Y]) - [X, Y] = 0 \tag{23}$$

for arbitrary vector fields X, Y .

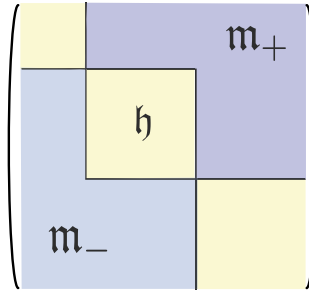


Fig. 2. The decomposition (20) of the Lie algebra.

- Using vector fields: the commutator of two holomorphic vector fields should be holomorphic, i.e.

$$(1 - i \mathcal{J})[(1 + i \mathcal{J})X, (1 + i \mathcal{J})Y] = 0. \tag{24}$$

(The property (24) may also be stated as the condition that the distribution of holomorphic vector fields is integrable.)

- Using forms: the holomorphic forms should form a differential ideal, i.e. the following condition should be satisfied: $d(J_-)_a \sim \sum_b R_{ab} \wedge (J_-)_b$ for some one-forms R_{ab} .

Let us demonstrate, that the last definition implies

$$[m_+, m_+] \subset m_+, \quad [m_-, m_-] \subset m_-, \tag{25}$$

if the restriction to \mathfrak{m} of the adjoint-invariant metric \mathbb{G} on \mathfrak{u}_N is Hermitian w.r.t. the chosen almost complex structure \mathcal{J} . In general the integrability of an almost complex structure means that $[m_+, m_+] \subset m_+ \oplus \mathfrak{h}$. To see this, note that an almost complex structure \mathcal{J} is defined by the conditions $\mathcal{J} \circ J_{\pm} = \pm i J_{\pm}$, where J_{\pm} are the components of a Maurer–Cartan current:

$$J = -g^{-1}dg = J_{\mathfrak{h}} + J_+ + J_-, \quad J_{\pm} \in m_{\pm}. \tag{26}$$

Since $dJ - J \wedge J = 0$, we get

$$dJ_- = [-J_0 \wedge J_0 + (\text{terms with } J_-) - J_+ \wedge J_+]_{m_-}.$$

Therefore for the integrability of \mathcal{J} one should have $[J_+ \wedge J_+]_{m_-} = 0$, i.e. $[m_+, m_+] \subset m_+ \oplus \mathfrak{h}$. We see that the conditions (25) therefore define an integrable complex structure. Conversely suppose we have an integrable complex structure on G/H , and m_{\pm} are its respective holomorphic/anti-holomorphic subspaces. Then $[a, b] = c + \gamma$, where $a, b, c \in m_+$ and $\gamma \in \mathfrak{h}$. Since $\langle m_+, \mathfrak{h} \rangle_{\mathbb{G}} = 0$, computing the scalar product with a generic element $\gamma' \in \mathfrak{h}$, we obtain $\langle \gamma', [a, b] \rangle_{\mathbb{G}} = \langle \gamma', \gamma \rangle_{\mathbb{G}}$. Using the identity $\langle [a, \gamma'], b \rangle_{\mathbb{G}} + \langle \gamma', [a, b] \rangle_{\mathbb{G}} = 0$, we get $\langle \gamma', \gamma \rangle_{\mathbb{G}} = -\langle [a, \gamma'], b \rangle_{\mathbb{G}} = \langle a', b \rangle_{\mathbb{G}}$, and $a' = [\gamma', a] \in m_+$. As discussed earlier, the subspace m_+ is isotropic, if the metric \mathbb{G} is Hermitian, therefore $\langle \gamma', \gamma \rangle_{\mathbb{G}} = 0$ for all $\gamma' \in \mathfrak{h}$, which implies $\gamma = 0$ due to the non-degeneracy of \mathbb{G} . The result $[m_+, m_+] \subset m_+$ follows.

A typical integrable complex structure on the flag manifold defines the holomorphic/anti-holomorphic subspaces m_{\pm} shown in Fig. 2.

As discussed above, one can define an *almost complex structure* on \mathcal{F}_N by choosing $\frac{N(N-1)}{2}$ mutually non-conjugate forms $J_{p_1q_1}, \dots, J_{m \frac{N(N-1)}{2} n \frac{N(N-1)}{2}}$ and declaring them holomorphic. The

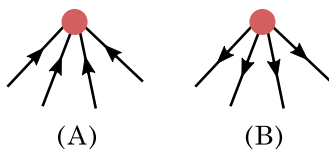


Fig. 3. (A) The ‘sink’ vertex, (B) The ‘source’ vertex.

remaining $\frac{N(N-1)}{2}$ forms will be therefore anti-holomorphic. To determine, which of those complex structures are integrable, it is useful to use a diagrammatic representation. We draw N vertices, as well as arrows from the node p_1 to the node q_1 , from p_2 to q_2 and so on (such diagrams are called ‘tournaments’, see [48]). The integrability of the almost complex structure, defined in this way, is equivalent to the acyclicity of the graph (the condition that it should not contain closed cycles). For a proof see [33], for example.

One can now establish the following fact:

Lemma. *There are exactly $N!$ acyclic diagrams.*

Proof. The statement of the lemma implies that there is only one combinatorial type of diagrams, and all acyclic diagrams may be obtained from any one of them by the action of the permutation group S_N . Let us describe this combinatorial type. Every acyclic diagram has a ‘source’-vertex, in which all the lines are outgoing, and a ‘sink’-vertex, in which all lines are incoming (see Fig. 3). Indeed, if that were not so, every vertex would contain at least, say, one outgoing line. Then one can start at any vertex and follow outgoing lines, until a loop is formed. Let us consider the ‘source’-vertex. The diagram formed by the remaining $N - 1$ vertices together with the edges joining them can be an arbitrary acyclic diagram (as the chosen vertex is a ‘source’, there cannot be cycles containing it). Therefore we have performed the first step of the induction. The subsequent steps consist in finding the ‘source’ vertex in the reduced diagram. It is therefore clear that there always exists a vertex with i outgoing lines for all $i = 0, \dots, N - 1$. This statement completely describes the combinatorial structure of the diagram. Equivalently, there is a total ordering on the set of vertices. Different diagrams differ just by a relabeling of the vertices.

We have established that there are $N!$ complex structures on a complete flag manifold $\frac{U(N)}{U(1)^N}$. Analogously there are $m!$ complex structures on a partial flag manifold $\frac{U(N)}{U(n_1) \times \dots \times U(n_m)}$. The number of complex structures may be easily interpreted as follows. Choosing a complex structure is equivalent to choosing a complex quotient space representation (2). In order to construct such representation one should choose a sequence of embedded linear spaces of the type (1), and the dimensions of these spaces are given by the partial sums of the integers n_i . These dimensions are therefore determined by an ordering of the set $\{n_i\}$, and there are $m!$ such orderings. \square

1.4. Cohomology

The second cohomology group of the complete flag manifold is

$$H^2(\mathcal{F}_N, \mathbf{Z}) = \mathbf{Z}^{N-1}, \tag{27}$$

hence there exist $N - 1$ linearly independent 2-forms, which are the generators of $H^2(\mathcal{F}_N)$. As a model for $H^2(\mathcal{F})$ let us consider the following construction. On \mathcal{F}_N there are N standard line bundles L_1, \dots, L_N , moreover their sum is a trivial bundle:

$$\bigoplus_{i=1}^N L_i = \mathcal{F}_N \times \mathbb{C}^N. \tag{28}$$

The first Chern classes of these bundles give N closed 2-forms: $\Omega_i = c_1(L_i)$, $i = 1 \dots N$. Due to the condition (28) and the additivity of the first Chern classes $c_1(E \oplus F) = c_1(E) + c_1(F)$ it is clear that the forms Ω_i are not independent but rather satisfy the relation

$$\sum_{i=1}^N \Omega_i = 0 \tag{29}$$

The two-forms Ω_i , $i = 1 \dots N$, satisfying the relation (29), generate $H^2(\mathcal{F}_N, \mathbb{Z})$.

One can also obtain an even more explicit description of the cohomology, which will shed some light on the relation (29). To this end we note that there exists an embedding

$$i : \mathcal{F}_N \hookrightarrow \underbrace{\mathbb{C}\mathbb{P}^{N-1} \times \dots \times \mathbb{C}\mathbb{P}^{N-1}}_{N \text{ factors}}. \tag{30}$$

A point $m \in (\mathbb{C}\mathbb{P}^{N-1})^{\times N}$ is a collection of N lines in \mathbb{C}^N through the origin. Those points that correspond to N pairwise orthogonal lines are the points lying on the flag manifold \mathcal{F}_N : here one should recall that \mathcal{F}_N may be thought of as the manifold of sets of N ordered mutually orthogonal lines in \mathbb{C}^N . Let us now consider $(\mathbb{C}\mathbb{P}^{N-1})^{\times N}$ as a symplectic manifold with a product symplectic form

$$\omega = \sum_{i=1}^N \omega_i \tag{31}$$

One can check (this is proven in [41]) that (30) is a Lagrangian embedding w.r.t. this symplectic form, i.e.

$$\omega|_{\mathcal{F}_N} = 0. \tag{32}$$

Identifying $\Omega_i = [i^*(\omega_i)]$ and taking into account (32), we obtain the relation (29). One can analogously prove that a general flag manifold, embedded into a product of Grassmannians $\mathcal{F}_{n_1, \dots, n_m} \hookrightarrow G_{n_1, N} \times \dots \times G_{n_m, N}$, is also a Lagrangian submanifold.

The above observations are tightly related to the theory of geometric quantization, i.e. to the construction of the representation theory of a Lie group G using line bundles over its flag manifolds. See our paper [41] for a review.

2. The models

In this paper we will continue our study of σ -models proposed in [31]. The target spaces of these models are homogeneous complex target spaces $\mathcal{M} = \frac{G}{H}$, endowed with the ‘Killing metric’ \mathbb{G} and an integrable G -invariant complex structure \mathcal{J} . The action schematically takes the form:

$$\mathcal{S}[\mathbb{G}, \mathcal{J}] := \int_{\Sigma} d^2z \|\partial X\|_{\mathbb{G}}^2 + \int_{\Sigma} X^* \omega, \tag{33}$$

where X is the map $X : \Sigma \rightarrow \mathcal{M}$ and ω is the Kähler form corresponding to the pair $(\mathbb{G}, \mathcal{J})$, defined as $\omega = \mathbb{G} \circ \mathcal{J}$. The peculiarity of such models lies in the fact that one can construct zero-curvature representations for their equations of motion. More precisely, a one-parametric

family $\mathcal{A}_u, u \in \mathbb{C}$, of flat connections can be constructed from the Noether current K using the formula of Pohlmeyer [2]:

$$\mathcal{A}_u = \frac{1-u}{2} K_z dz + \frac{1-u^{-1}}{2} K_{\bar{z}} d\bar{z}. \tag{34}$$

In the present paper we define the complex coordinates z, \bar{z} as $\{z, \bar{z}\} = \frac{x^1 \pm ix^2}{\sqrt{2}}$, so that the line element is $ds^2 = 2 dz d\bar{z}$, and the metric is $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The connections defined in (34) are flat, since the Noether current K itself is flat for the models above. This construction is well-known for symmetric spaces [49], which e.g. for the group $G = SU(N)$ are the Grassmannians $G_{k,N}$. Our models allow for a richer class of target spaces, in particular including the so-called flag manifolds.

2.0.1. Dependence on the metric

Let us clarify that here and in what follows when referring to the Noether current, we mean a one-form K . In fact, the canonical Noether procedure produces a vector field K^μ , satisfying the conservation equation $\partial_\mu(\sqrt{g}K^\mu) = 0$. The one-form is obtained from this vector field by lowering the index in the standard way: $K = g_{\mu\nu}K^\nu dx^\mu$. The Hodge dual to this one-form is $*K = \frac{1}{\sqrt{g}}g_{\alpha\beta}\epsilon^{\beta\gamma} K_\gamma dx^\alpha$ and

$$d * K = \partial_\lambda \left(\frac{1}{\sqrt{g}} g_{\alpha\beta} \epsilon^{\beta\gamma} K_\gamma \right) dx^\lambda \wedge dx^\alpha = \partial_\lambda \left(\frac{1}{\sqrt{g}} \epsilon^{\lambda\alpha} g_{\alpha\beta} \epsilon^{\beta\gamma} K_\gamma \right) dx^1 \wedge dx^2 = \tag{35}$$

$$= -\frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g} g^{\lambda\gamma} K_\gamma) \text{vol}_{\Sigma_2}. \tag{36}$$

Let us consider a simple example of a free complex massless boson, the action being $S = \int dx^1 dx^2 \sqrt{g} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \bar{\psi}$. The corresponding one-form is then $K = i(\bar{\psi} d\psi - d\bar{\psi} \psi)$ and, in particular, it is independent of the worldsheet metric g . Suppose we now include a certain invariant B -field, or even a topological term, replacing the action by $\tilde{S} = S + \kappa \int i d\bar{\psi} \wedge d\psi$. The Noether current should now be replaced by $\tilde{K} = K + \kappa * d(\bar{\psi} \psi)$, the additional component being a topological current. Nevertheless, the full current \tilde{K} now depends on the conformal class of the metric g , through the dependence of the Hodge star $*$. We come to the conclusion that, in general, both one-forms K and $*K$ depend on the conformal class $[g]$ of the worldsheet metric. ■

In the present paper we will concentrate on the simplest non-symmetric (i.e. non-Grassmannian) example, when

$$\mathcal{F} := \frac{SU(N)}{S(U(n) \times U(m) \times U(N-m-n))}. \tag{37}$$

This manifold may be thought of as the space of mutually orthogonal (w.r.t. some metric on \mathbb{C}^N) linear subspaces E_n, E_m, E_{N-m-n} of \mathbb{C}^N , passing through the origin, of dimensions n, m and $N-m-n$. Clearly, these subspaces span all of \mathbb{C}^N .

3. Relation to models with \mathbb{Z}_m -graded target spaces

Although the manifold (37) is not a symmetric space, i.e. not a space with a \mathbb{Z}_2 -grading, it is a space with a \mathbb{Z}_3 -grading. Before explaining the concept, we note that natural examples of

\mathbb{Z}_3 -graded spaces are provided by twistor spaces of symmetric spaces [51] and nearly Kähler homogeneous spaces [52]. In this context the manifold (37) may be seen as the twistor space of $G_{n,N}$, $G_{m,N}$ or $G_{N-m-n,N}$.

A homogeneous space $\frac{G}{H}$ is called \mathbb{Z}_m -graded (or m -symmetric), if the Lie algebra \mathfrak{g} of its isometry group admits the following decomposition:

$$\mathfrak{g} = \bigoplus_{i=0}^{m-1} \mathfrak{g}_i, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \bmod m}, \quad \mathfrak{g}_0 = \mathfrak{h}. \tag{38}$$

In this language the ordinary symmetric spaces are 2-symmetric spaces. Then, one can show, that, similarly to what happens for symmetric spaces, the e.o.m. of certain σ -models with \mathbb{Z}_m -graded target-spaces may be rewritten as flatness conditions of a one-parametric family of connections.

A related class of models was considered in [24] and subsequently studied in [50]. The action studied in [24] has the form

$$\tilde{\mathcal{S}} := \int_{\Sigma} d^2x \|\partial X\|_G^2 + \int_{\Sigma} X^* \tilde{\omega}, \tag{39}$$

where $\tilde{\omega}$ is a 2-form constructed using the \mathbb{Z}_m -decomposition of the Lie algebra (38). Note that, if $\tilde{\omega}$ were the Kähler form, one would obtain precisely the action (33). Now we come to the precise definition of $\tilde{\omega}$. Decompose the current $J = -g^{-1}dg$ according to (38):

$$J = -g^{-1}dg = \sum_{i=0}^{m-1} J^{(i)}, \quad \text{where } J^{(i)} \in \mathfrak{g}_i. \tag{40}$$

The form $\tilde{\omega}$ is defined as follows:

$$\tilde{\omega} = \frac{1}{2} \sum_{k=1}^{m-1} \frac{(m-k)-k}{m} \text{tr}(J^{(k)} \wedge J^{(m-k)}) \tag{41}$$

This formula poses an important question. According to (41), the form $\tilde{\omega}$ depends on the \mathbb{Z}_m -grading on the Lie algebra. On the other hand, generally a given Lie algebra \mathfrak{g} may have many different gradings (with different, or same, values of m). The question is: are the models defined by (39)–(41), corresponding to different gradings of \mathfrak{g} , different?

Before answering this question, first we review the construction of cyclic gradings on semi-simple Lie algebras, which was completed long ago [53]. Let us consider, for simplicity, the case of $\mathfrak{g} = \mathfrak{su}(N+1)$. A cyclic grading may be constructed as follows¹: one picks a system of N simple positive roots $\alpha_1, \dots, \alpha_N$, as well as the maximal negative root $\alpha_{N+1} = -\alpha_1 - \dots - \alpha_N$.² Then one assigns to these $N+1$ roots arbitrary (non-negative integer) gradings m_1, \dots, m_N, m_{N+1} . The

¹ Here we restrict ourselves to the grading of type $A_N^{(1)}$.

² In the paper of Kac [53] the roots $\alpha_1, \dots, \alpha_{N+1}$ are seen as the positive simple roots of the corresponding affine Lie algebra \hat{A}_N . Consider the case $N = 2$. The simple positive roots of the loop algebra $\mathfrak{su}_3(t, t^{-1})$ may be chosen as follows:

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_0 = t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this context the latter root α_0 – the analog of α_{N+1} – is customarily called ‘imaginary’. In fact, the whole theory of cyclic Lie algebra gradings is formulated by Kac naturally in terms of affine Lie algebras and their Dynkin diagrams.

gradings of all other roots are determined by the Lie algebra structure, and the value of m is calculated as

$$m = m_1 + \dots + m_{N+1}. \tag{42}$$

In usual matrix form, this grading looks as follows:

$$\begin{pmatrix} 0 & \mathbf{m}_1 & & & \\ & 0 & \mathbf{m}_2 & & \\ & & 0 & \ddots & \\ & & & 0 & \mathbf{m}_N \\ \mathbf{m}_{N+1} & & & & 0 \end{pmatrix} \tag{43}$$

The subalgebra \mathfrak{g}_0 , which determines the denominator H of the quotient space G/H , is determined by those m_i 's, which are zero. For example, if all $m_i > 0$, the resulting space is the manifold of complete flags $\frac{SU(N+1)}{S(U(1)^{N+1})}$.

In general, for a choice of grading determined by the set m_1, \dots, m_{N+1} some of the subspaces \mathfrak{g}_i will be identically zero. Therefore a natural restriction to adopt is to require that $\mathfrak{g}_i \neq 0$ for all $i \pmod m$. We will call such a grading *admissible*. This still leaves a wide range of possibilities. For example, in the case of $SU(3)$ the following is a complete list of admissible gradings (up to the action of the Weyl group S_3):

$$\mathbb{Z}_2: \begin{pmatrix} 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}, \quad \mathbb{Z}_3: \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} \\ 2 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & 0 \end{pmatrix}, \tag{44}$$

$$\mathbb{Z}_4: \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} \\ 3 & 0 & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & 0 \end{pmatrix}, \quad \mathbb{Z}_5: \begin{pmatrix} 0 & \mathbf{1} & \mathbf{3} \\ 4 & 0 & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} \\ 4 & 0 & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & 0 \end{pmatrix}, \tag{45}$$

$$\mathbb{Z}_6: \begin{pmatrix} 0 & \mathbf{1} & \mathbf{3} \\ 5 & 0 & \mathbf{2} \\ \mathbf{3} & \mathbf{4} & 0 \end{pmatrix}, \quad \mathbb{Z}_7: \begin{pmatrix} 0 & \mathbf{1} & \mathbf{3} \\ 6 & 0 & \mathbf{2} \\ \mathbf{4} & \mathbf{5} & 0 \end{pmatrix} \tag{46}$$

The \mathbb{Z}_2 -grading and the second \mathbb{Z}_3 -grading correspond to the homogeneous space $SU(3)/S(U(2) \times U(1)) = \mathbb{C}P^2$, and all other gradings correspond to the flag manifold $\mathcal{F}_{1,1,1}$.

We will now give an answer to the question posed above: what is the relation between the σ -models with the action (39), taken for different gradings on the corresponding Lie algebra? Our statement is [35]:

Proposition. *For homogeneous spaces of the unitary group, the models defined by (39)–(41) with different $A_N^{(1)}$ -type gradings on \mathfrak{g} are classically equivalent to the model defined by the action (33) with some choice of complex structure on the target-space.*

In fact, one has a precise statement about the relation of the B -fields in the two models. To formulate it, we ‘solve’ the constraint (42) as follows³:

³ Formula (47) implies that the cyclic automorphism $\widehat{\Omega}$ of the Lie algebra, which defines the \mathbb{Z}_m grading, can be represented as follows: $\widehat{\Omega}(a) = \Omega a \Omega^{-1}$, where $\Omega = \text{diag}(e^{2\pi i \frac{n_1}{m}}, \dots, e^{2\pi i \frac{n_{N+1}}{m}})$.

$$m_k = n_k - n_{k+1} \tag{47}$$

where n_k are integers and $n_{N+2} \equiv n_1 - m$. We then have (see [35] for a proof):

$$\tilde{\omega} = \omega - 2 \sum_{i=1}^N \frac{n_i}{m} dJ_{ii} \tag{48}$$

We see that, irrespective of the choice of the grading (which is now encoded in the integers n_i), the form $\tilde{\omega}$ differs from the Kähler form by a topological term. This topological term, clearly, depends on the chosen grading, but does not contribute to the equations of motion.

Comment. Although the flag manifolds (4) are \mathbb{Z}_m -graded spaces, the two classes of target spaces – \mathbb{Z}_m -graded and complex homogeneous spaces – do not coincide. For example, one has the space $\frac{G_2}{SU(3)} \simeq S^6$. The stability subgroup $SU(3)$ acts on the tangent space $\mathfrak{m} = \mathbb{R}^6$ via $V \oplus \bar{V}$, where $V \simeq \mathbb{C}^3$ is the standard representation. Therefore it has a unique almost complex structure, which is *not* integrable (see the review [54]). On the other hand, it is a nearly Kähler manifold and is \mathbb{Z}_3 -graded [52]. On the other side of the story, one has the complex manifold $S^1 \times S^3 \simeq U(2)$ (see [35] for a discussion), which may be viewed as a \mathbb{T}^2 -bundle over $\mathbb{C}P^1$ (the simplest flag manifold). This manifold is *not* a \mathbb{Z}_m -graded homogeneous space of the group $G = U(2)$.

We also note that the construction of Lax connections for models with \mathbb{Z}_m -graded spaces was explored in [50]. The fact that the corresponding integrals of motion are in involution was proven, for instance, in [30].

4. The GLSM representation

In this section we will construct a gauged linear sigma-model (GLSM) representation for the models (33) introduced earlier (for flag manifold target spaces \mathcal{F}). We recall that these models depend explicitly on the complex structure \mathcal{J} . Now, picking a complex structure \mathcal{J} on \mathcal{F} is equivalent to picking a total ordering of the subspaces E_n, E_m, E_{N-m-n} . For the purposes of constructing a $\frac{1}{N}$ -expansion any of the two orderings, where E_{N-m-n} is maximal, will suffice (the reason will become clear shortly). Let us prove, first of all, that it is always possible to achieve this, without loss of generality. The point here is that the action (33), albeit depending on the complex structure, might produce the same equations of motion even for different choices of complex structure. This is due to the fact, that for certain complex structures, which we denote by \mathcal{J}_1 and \mathcal{J}_2 , the difference in the two actions may just be a topological term:

$$\mathcal{S}[\mathcal{J}_1] - \mathcal{S}[\mathcal{J}_2] = \int_{\Sigma} \theta_{12}, \quad d\theta_{12} = 0. \tag{49}$$

Let us describe precisely the situation when this happens. To this end we recall that, as was established at the end of section 1.3, the complex structures are in a one-to-one correspondence with an ordering of the mutually orthogonal spaces $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_m}$, that are a point in a flag manifold $\frac{U(N)}{U(n_1) \times \dots \times U(n_m)}$.

Proposition. *The actions $\mathcal{S}[\mathcal{J}_1]$ and $\mathcal{S}[\mathcal{J}_2]$ differ by a topological term, as in (49), if and only if the corresponding sequences of spaces $\{\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_m}\}$ differ by a cyclic permutation.*

Proof. Let us call \mathcal{J} the standard complex structure, whose holomorphic subspace \mathfrak{m}_+ is given by upper-block-triangular matrices. Then $\mathcal{J}_1 = \sigma_1(\mathcal{J})$ and $\mathcal{J}_2 = \sigma_2(\mathcal{J})$ for some permutations $\sigma_1, \sigma_2 \in S_m$. We recall the notation J_{ij} from (9). The corresponding Kähler forms are

$$\omega_1 = i \sum_{i < j} \text{Tr}(J_{\sigma_1(i)\sigma_1(j)} \wedge J_{\sigma_1(j)\sigma_1(i)}) \tag{50}$$

$$\omega_2 = i \sum_{i < j} \text{Tr}(J_{\sigma_2(i)\sigma_2(j)} \wedge J_{\sigma_2(j)\sigma_2(i)}) \tag{51}$$

Upon introducing the notation $\sigma(J_{ij}) := J_{\sigma(i)\sigma(j)}$, we may write the difference of the two forms as

$$\omega_1 - \omega_2 = i \sigma_1 \left(\sum_{i < j} \text{Tr}(J_{ij} \wedge J_{ji}) - \sum_{i < j} \text{Tr}(J_{\tau^{-1}(i)\tau^{-1}(j)} \wedge J_{\tau^{-1}(j)\tau^{-1}(i)}) \right), \tag{52}$$

where $\tau^{-1} = \sigma_1^{-1}\sigma_2$. This reduces the problem to that of $\mathcal{J}_1 = \mathcal{J}$ and $\mathcal{J}_2 = \tau^{-1}(\mathcal{J})$. Note that the exterior derivative commutes with the permutation σ , due to the following simple fact following from the Maurer–Cartan equation: $\sigma(dJ_{ij}) = \sigma(\sum_k J_{ik} \wedge J_{kj}) = \sum_k \sigma(J_{ik}) \wedge \sigma(J_{kj}) = d\sigma(J_{ij})$, where to arrive at the last equality one has to make a change of the dummy summation index $k \rightarrow \sigma(k)$.

We wish to show that $d(\omega_1 - \omega_2) = 0$ implies that τ is a cyclic permutation. To this end we rewrite the above difference as follows:

$$\omega_1 - \omega_2 = i \sigma_1 \left(\sum_{i,j} \alpha_{ij} \text{Tr}(J_{ij} \wedge J_{ji}) \right), \tag{53}$$

$$\text{where } \alpha_{ij} = \frac{1}{2}(\text{sgn}(j - i) - \text{sgn}(\tau(j) - \tau(i))) \in \{-1, 0, 1\}. \tag{54}$$

(We have made a change of dummy variables $i \rightarrow \tau(i)$ and $j \rightarrow \tau(j)$ in the second sum in (52).) Closedness of this form requires that (see (12)–(13))

$$\alpha_{ij} = \frac{1}{2}(\text{sgn}(j - i) - \text{sgn}(\tau(j) - \tau(i))) = z_i - z_j. \tag{55}$$

Let us consider the case $j > i$. Then $z_i = z_j$ if $\tau(j) > \tau(i)$ and $z_i = z_j + 1$ if $\tau(j) < \tau(i)$. This means that $\{z_i\}_{i=1\dots m}$ form a non-increasing sequence, and moreover the difference between any two elements is either zero or 1. This is only possible if the set has the form $(\underbrace{z, \dots, z}_K, \underbrace{z - 1, \dots, z - 1}_{m-K})$. Accordingly the original sequence of m consecutive numbers can be split into two consecutive sets:

$$1 \dots m = (I_1, I_2). \tag{56}$$

Since $\tau(j) < \tau(i)$ for $(i \leq K, j > K)$, the permutation acts as follows:

$$\tau(I_1, I_2) = (\tau(I_2), \tau(I_1)). \tag{57}$$

Moreover, since $\tau(i) < \tau(j)$ for $i < j \leq K$ and the image $\tau(I_1)$ is $(m - K + 1, \dots, m)$, a moment’s thought shows that $\tau(i) = m - K + i$ for $i = 1 \dots K$. Analogously $\tau(j) = j - K$

for $j = K + 1 \dots m$. Therefore τ is nothing but a K -fold cyclic permutation ‘to the left’ (or $m - K$ -fold to the right).

Since for $i < j$ the non-zero α_{ij} are the ones, for which $\tau(j) < \tau(i)$, this implies $j = K + 1 \dots m$ and $i = 1 \dots K$. These α_{ij} are equal to 1, therefore

$$\omega_1 - \omega_2 = i \sigma_1 \left(\sum_{\substack{i=1 \dots K, \\ j=K+1 \dots n}} \text{Tr}(J_{ij} \wedge J_{ji}) \right), \tag{58}$$

which is easily seen to be proportional to the (generalized) Fubini–Study form on the Grassmanian $G_{L,N}$, where $L = \sum_{i=1}^K n_i$. Conversely, one shows that for a cyclic permutation the difference between ω_1 and ω_2 is the closed form written above. \square

Returning to the manifold (37), we may assume that we have picked an ordering of E_n, E_m, E_{N-m-n} , in which E_{N-m-n} is maximal (by making a cyclic permutation if necessary). The manifold \mathcal{F} with the complex structure defined by this ordering may then be thought of as the space of embedded linear complex spaces:

$$\mathcal{F} = \left\{ 0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^N \right\}, \tag{59}$$

where $V_1 \simeq \mathbb{C}^n$ and $V_2 \simeq \mathbb{C}^{m+n}$. As shown in our paper [34], the model (33) can be formulated in this case as a gauged linear σ -model, in terms of a ‘matter field’ $\varphi \in \text{Hom}(V_2, \mathbb{C}^N)$ and an auxiliary field $\mathcal{A} = \mathcal{A}_z dz + \mathcal{A}_{\bar{z}} d\bar{z} \in \mathfrak{u}_{m+n} \otimes T^* \Sigma$:

$$\mathcal{L} = \text{Tr} \left((\mathcal{D}_z \varphi)^\dagger (\mathcal{D}_z \varphi) \right) + i \text{Tr} \left(\lambda \left(\varphi^\dagger \varphi - \frac{N}{g^2} \mathbb{1}_{m+n} \right) \right) \tag{60}$$

The field φ is subject to a normalization constraint $\varphi^\dagger \varphi = \frac{N}{g^2} \mathbb{1}_{m+n}$, which is imposed by the Lagrange multiplier λ . Here g is the coupling constant of the model and the dependence on N is chosen to suit the $\frac{1}{N}$ -expansion, constructed in the next section. To simplify the situation even further, we will restrict to the case $m = n = 1$, i.e. we will consider the target space of the form

$$\mathcal{F} = \frac{U(N)}{U(1) \times U(1) \times U(N-2)}. \tag{61}$$

In this case we can express the matrix φ in terms of its two column vectors $\varphi = (u, v)$, then the normalization constraint says that $\|u\| = \|v\| = \frac{\sqrt{N}}{g}$ and $\bar{u} \circ v = 0$. The covariant derivatives in the above Lagrangian are defined in the standard way:

$$\mathcal{D}_z \varphi = \partial_z \varphi - i \varphi \mathcal{A}_z, \quad (\mathcal{D}_z \varphi)^\dagger = \partial_{\bar{z}} \varphi^\dagger + i \mathcal{A}_{\bar{z}} \varphi^\dagger. \tag{62}$$

Note that we assume the connection \mathcal{A} to be Hermitian, meaning that $\mathcal{A}^\dagger = \mathcal{A}$, hence $(\mathcal{A}_z)^\dagger = \mathcal{A}_{\bar{z}}$. An important point is that the ‘connection’ \mathcal{A} in the model above has to be taken in a non-conventional triangular form:

$$\mathcal{A}_z = \begin{pmatrix} a_z & 0 \\ c_z & b_z \end{pmatrix}, \quad \mathcal{A}_{\bar{z}} = (\mathcal{A}_z)^\dagger = \begin{pmatrix} a_{\bar{z}} & c_{\bar{z}} \\ 0 & b_{\bar{z}} \end{pmatrix} \tag{63}$$

The form of \mathcal{A} is the main difference of our models from the models with symmetric target spaces. In particular, this triangular form implies that there is only the $U(1)^2$ gauge symmetry,

for which a and b are the gauge fields, whereas c is another auxiliary field transforming in the bi-fundamental representation of $U(1)^2$. Had we not imposed the constraints (63), we would have the gauge group $U(2)$, and the target space of the model would be the Grassmannian $G_{2,N}$.

Integrating out the auxiliary fields a, b, c we get the following form of the Lagrangian (for details see the Appendix B):

$$\mathcal{L} = \partial_{\bar{z}}\bar{u} \circ \left(\mathbb{1}_N - \frac{g^2}{N} (u \otimes \bar{u} + v \otimes \bar{v}) \right) \circ \partial_z u + \partial_{\bar{z}}\bar{v} \circ \left(\mathbb{1}_N - \frac{g^2}{N} v \otimes \bar{v} \right) \circ \partial_z v. \tag{64}$$

This expression may be simplified, if we introduce an additional matrix-valued field $w : \mathbb{C}^{N-2} \rightarrow \mathbb{C}^N$, defined by the completeness relation in \mathbb{C}^N :

$$u \otimes u^\dagger + v \otimes v^\dagger + \text{Tr}_{\mathbb{C}^{N-2}}(w \otimes w^\dagger) = \frac{N}{g^2} \mathbb{1}_N. \tag{65}$$

This relation simply means that the matrix w contains $N - 2$ vectors, which are orthogonal to u and v and mutually orthonormal. We then obtain

$$\frac{N}{g^2} \mathcal{L} = \text{Tr} \left((\partial_{\bar{z}}\bar{u} \circ w)(w^\dagger \circ \partial_z u) \right) + \text{Tr} \left((\partial_{\bar{z}}\bar{v} \circ w)(w^\dagger \circ \partial_z v) \right) + (\partial_{\bar{z}}\bar{v} \circ u)(\bar{u} \circ \partial_z v). \tag{66}$$

This is precisely the type of Lagrangian considered in [31], [32], [33]. In this way we have established the equivalence of the model (33) with its gauged linear representation (60).

Before proceeding further, let us make one more observation regarding the Lagrangian (60). A direct calculation shows that the difference

$$\mathcal{L} - \tilde{\mathcal{L}} = \text{Tr} \left((\mathcal{D}_z \varphi)^\dagger (\mathcal{D}_z \varphi) \right) - \text{Tr} \left((\mathcal{D}_{\bar{z}} \varphi)^\dagger (\mathcal{D}_{\bar{z}} \varphi) \right) = \tag{67}$$

$$= \text{Tr} \left(\partial_z \varphi^\dagger \partial_z \varphi - \partial_{\bar{z}} \varphi^\dagger \partial_{\bar{z}} \varphi \right) \sim \text{the pull-back of } i \text{Tr} \left(d\varphi \wedge d\varphi^\dagger \right) \tag{68}$$

The last line is clearly a total derivative, being the pull-back to the worldsheet of a Fubini–Study form on the Grassmannian $G_{2,N}$ (with the normalization $\varphi^\dagger \varphi = \frac{N}{g^2} \mathbb{1}_2$ the Fubini–Study form is $\frac{N}{g^2} \Omega_{FS} \sim i \text{Tr} (d\varphi \wedge d\varphi^\dagger)$). Taking advantage of this fact, we will replace the Lagrangian (60) by a more symmetric one

$$\mathcal{L} = \text{Tr}(\mathcal{D}_\mu \varphi^\dagger \mathcal{D}_\mu \varphi) + i \text{Tr} \left(\lambda \left(\varphi^\dagger \varphi - \frac{N}{g^2} \mathbb{1}_2 \right) \right) \tag{69}$$

Above it was claimed that the Noether current K derived from the action (33) is flat, i.e. $dK - K \wedge K = 0$. Let us check this explicitly. To start with, the Noether current has the form

$$K = \frac{2g^2}{N} \left(\varphi (\mathcal{D}_z \varphi)^\dagger d\bar{z} - \mathcal{D}_{\bar{z}} \varphi \varphi^\dagger dz \right). \tag{70}$$

Note that $K \in \mathfrak{u}_N$ and therefore $K^\dagger = -K$. Now,

$$\begin{aligned} dK - K \wedge K &= \frac{2g^2}{N} \left(2 \mathcal{D}_z \varphi (\mathcal{D}_z \varphi)^\dagger + \varphi \mathcal{D}_z \mathcal{D}_{\bar{z}} \varphi^\dagger + \mathcal{D}_{\bar{z}} \mathcal{D}_z \varphi \varphi^\dagger - \right. \\ &\quad \left. - \frac{2g^2}{N} \mathcal{D}_z \varphi (\varphi^\dagger \varphi) \mathcal{D}_{\bar{z}} \varphi^\dagger + \frac{2g^2}{N} \varphi (\mathcal{D}_{\bar{z}} \varphi^\dagger \mathcal{D}_z \varphi) \varphi^\dagger \right) dz \wedge d\bar{z} \end{aligned} \tag{71}$$

Let us now simplify this expression, using the normalization condition $\varphi^\dagger \varphi = \frac{N}{g^2} \mathbb{1}_2$, as well as the equations of motion following from the Lagrangian (69):

$$\begin{aligned} \mathcal{D}_{\bar{z}}\mathcal{D}_z\varphi &= \frac{1}{2}\varphi(i\lambda + iF_{z\bar{z}}), & \mathcal{D}_z\mathcal{D}_{\bar{z}}\varphi^\dagger &= \frac{1}{2}(i\lambda + iF_{z\bar{z}})\varphi^\dagger \\ \varphi^\dagger\mathcal{D}_z\varphi &= -\mathcal{D}_z\varphi^\dagger\varphi = \begin{pmatrix} 0 & \bullet \\ 0 & 0 \end{pmatrix}, & \mathcal{D}_{\bar{z}}\varphi^\dagger\varphi &= -\varphi^\dagger\mathcal{D}_{\bar{z}}\varphi = \begin{pmatrix} 0 & 0 \\ \bullet & 0 \end{pmatrix} \end{aligned} \tag{72}$$

(we have used the identities $[D_z, D_{\bar{z}}]\varphi = -i\varphi F_{z\bar{z}}, [D_z, D_{\bar{z}}]\varphi^\dagger = iF_{z\bar{z}}\varphi^\dagger$).

Note an important difference with the standard case of symmetric target spaces (Grassmannians), where one has $\varphi^\dagger\mathcal{D}_z\varphi = \mathcal{D}_{\bar{z}}\varphi^\dagger\varphi = 0$.

It follows from the two equations in (72) that $i\lambda$ is Hermitian: $(i\lambda)^\dagger = i\lambda$. On the other hand, multiplying the first equation by φ^\dagger from the left, we get

$$\frac{1}{2}i\lambda = \frac{g^2}{N}\varphi^\dagger\mathcal{D}_{\bar{z}}\mathcal{D}_z\varphi - \frac{1}{2}iF_{z\bar{z}} = \frac{g^2}{N}\mathcal{D}_{\bar{z}}(\varphi^\dagger\mathcal{D}_z\varphi) - \frac{g^2}{N}\mathcal{D}_{\bar{z}}\varphi^\dagger\mathcal{D}_z\varphi - \frac{1}{2}iF_{z\bar{z}} \tag{73}$$

The last two terms in the r.h.s. are manifestly Hermitian, whereas the first term turns out to be upper-triangular:

$$\mathcal{D}_{\bar{z}}(\varphi^\dagger\mathcal{D}_z\varphi) = \begin{pmatrix} 0 & \tilde{\bullet} \\ 0 & 0 \end{pmatrix}, \quad \tilde{\bullet} = (\partial_{\bar{z}} + i(a_{\bar{z}} - b_{\bar{z}}))\bar{u} \circ \partial_z v. \tag{74}$$

Hermiticity of $i\lambda$ then implies that $\mathcal{D}_{\bar{z}}(\varphi^\dagger\mathcal{D}_z\varphi) = 0$ and therefore

$$\frac{1}{2}i\lambda = -\frac{g^2}{N}\mathcal{D}_{\bar{z}}\varphi^\dagger\mathcal{D}_z\varphi - \frac{1}{2}iF_{z\bar{z}}. \tag{75}$$

Using all this information in (71), we find that $dK - K \wedge K = 0$, as claimed earlier.

4.1. General flag manifolds

Although the calculations in the following sections will concern the case of the target space (61), for completeness let us introduce the GLSM-formulation in the general case of the flag manifolds $\frac{U(N)}{U(n_1) \times \dots \times U(n_m)}, \sum_{i=1}^m n_i = N$. We assume that an arbitrary complex structure \mathcal{J} has been chosen, which is tantamount to choosing an ordering of the factors

$$U(n_1) \times \dots \times U(n_m), \tag{76}$$

as was explained in Section 1.3. We can always reshuffle n_1, \dots, n_m in such a way that the ordering is the literal ordering of factors in (76). In this case we are dealing with the manifold of complex flags

$$0 \subset V_1 \subset \dots \subset V_m = \mathbb{C}^N, \quad \dim_{\mathbb{C}} V_k = \sum_{i=1}^k n_i. \tag{77}$$

We define $M := \dim V_{m-1}$ and introduce the matter field $\varphi \in \text{Hom}(\mathbb{C}^M, \mathbb{C}^N)$. Its columns parametrize M vectors that define the flag. They are orthonormal:

$$\varphi^\dagger\varphi = \frac{N}{g^2}\mathbb{1}_M. \tag{78}$$

This is an analog of the moment map constraint from the theory of Kähler quotients. Now we introduce an analogue of the gauge field

$$A = \mathcal{A}_z dz + \mathcal{A}_{\bar{z}} d\bar{z}, \tag{79}$$

$$\mathcal{A}_z := \begin{pmatrix} (A_{11})_z & 0 & 0 & \cdots & 0 \\ (A_{21})_z & (A_{22})_z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (A_{m-11})_z & (A_{m-12})_z & \cdots & \cdots & (A_{m-1m-1})_z \end{pmatrix}, \quad \mathcal{A}_{\bar{z}} = (\mathcal{A}_z)^\dagger \tag{80}$$

and the covariant derivative

$$\mathcal{D}_\mu \varphi := \partial_\mu \varphi - i \varphi \mathcal{A}_\mu. \tag{81}$$

In (80) the diagonal blocks $(A_{jj})_z$ are of size $n_j \times n_j$ and represent gauge fields for the respective gauge groups $U(n_j)$. The off-diagonal blocks $(A_{ij})_z, i \neq j$ are additional ‘matter fields’ transforming in bi-fundamental representations of the corresponding gauge groups $U(n_i) \times U(n_j)$. Note that the gauge group $U(n_m)$ does not appear in the above formulas – this is analogous to the situation familiar from the $\mathbb{C}P^{N-1}$ -model, where the standard GLSM-representation involves only the $U(1)$ gauge field, despite the fact that the quotient space form $\mathbb{C}P^{N-1} \simeq \frac{U(N)}{U(1) \times U(N-1)}$ involves a product of gauge groups $U(1) \times U(N-1)$.

The Lagrangian of the model is a direct generalization of (69):

$$\mathcal{L} = \text{Tr}((\mathcal{D}_\mu \varphi)^\dagger \mathcal{D}_\mu \varphi) + \text{Tr} \left(\lambda \left(\varphi^\dagger \varphi - \frac{N}{g^2} \mathbb{1}_M \right) \right). \tag{82}$$

As a result, we have obtained a theory that is very similar to the Grassmannian $G_{M,N}$ sigma-model, but with a ‘reduced’ gauge field.

5. Feynman rules for the $\frac{1}{N}$ -expansion

We now return to the case of the target space (61). In the previous section we introduced the Lagrangian (69) of our model. The Lagrangian in the form (60) is not the most convenient one for carrying out the $\frac{1}{N}$ -expansion, as it would lead to mixed $\langle \mathcal{A} \lambda \rangle$ propagators. For this reason we will be using the Lagrangian in the form (69):

$$\mathcal{L} = \text{Tr}(\mathcal{D}_\mu \varphi^\dagger \mathcal{D}_\mu \varphi) + i \text{Tr}(\lambda (\varphi^\dagger \varphi - \frac{N}{g^2} \mathbb{1}_2)) \tag{83}$$

One should keep in mind the normalization of the metric that leads to $D_\mu \otimes D_\mu = D_{\bar{z}} \otimes D_z + D_z \otimes D_{\bar{z}}$. Note that we have also normalized the fields differently now:

$$\varphi^\dagger \varphi = \frac{1}{\hat{g}^2} \mathbb{1}_2 = \frac{N}{g^2} \mathbb{1}_2. \tag{84}$$

This is the proper normalization for carrying out the $\frac{1}{N}$ -expansion. We see that in the large- N limit the coupling constant $\hat{g} \sim \frac{1}{\sqrt{N}}$ vanishes, just like in the large- N limit of gauge theories. The renormalized coupling constant g stays fixed in the large- N limit.

Quantization of the theory (83) as it stands will lead to one-loop divergences. To get rid of them, we will introduce additional fermionic ghost fields χ – the Pauli–Villars regulators:

$$\begin{aligned} \widehat{\mathcal{L}} = & \text{Tr} \left((\mathcal{D}_\mu \varphi)^\dagger (\mathcal{D}_\mu \varphi) \right) + \text{Tr} \left((\mathcal{D}_\mu \chi)^\dagger (\mathcal{D}_\mu \chi) \right) + M^2 \text{Tr}(\chi^\dagger \chi) + \\ & + i \text{Tr} \left(\Lambda (\varphi^\dagger \varphi + \chi^\dagger \chi - \frac{N}{g^2} \mathbb{1}_2) \right). \end{aligned} \tag{85}$$

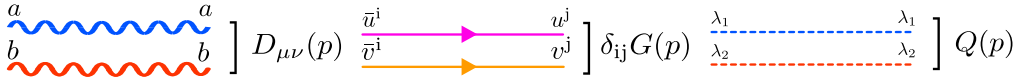


Fig. 4. The propagators of the (doubled) $\mathbb{C}\mathbb{P}^{N-1}$ model.

Integrating over the ‘matter fields’ φ, χ in the path integral, we obtain the effective action:

$$\mathcal{S}^{\text{eff}} = -N \text{Tr} \text{Log}(-\mathcal{D}_\mu \mathcal{D}_\mu + i\Lambda) + N \text{Tr} \text{Log}(-\mathcal{D}_\mu \mathcal{D}_\mu + M^2 + i\Lambda) - i \frac{N}{g^2} \int d^2z \text{Tr} \Lambda. \tag{86}$$

We will now find the saddle point w.r.t. Λ , assuming that at the saddle point $\Lambda_0 = -i m^2 \mathbb{1}_2$ and $\langle \mathcal{A}_z \rangle = \langle \mathcal{A}_{\bar{z}} \rangle = 0$:

$$\int d^2p \left(\frac{1}{p^2 + m^2} - \frac{1}{p^2 + M^2 + m^2} \right) = \frac{1}{g^2}. \tag{87}$$

This has the solution

$$m^2 = \frac{M^2 e^{-\frac{1}{g^2}}}{1 - e^{-\frac{1}{g^2}}}. \tag{88}$$

At the end of the day we wish to get rid of the ghost fields χ , so we will take the limit $M \rightarrow \infty$, in such a way that the mass m of the field φ stays finite, therefore in this limit we would need to take $g \rightarrow 0_+$ accordingly.

The Lagrangian (85) may be rewritten in the form of two coupled $\mathbb{C}\mathbb{P}^{N-1}$ -model Lagrangians:

$$\begin{aligned} \widehat{\mathcal{L}} = & |D_\mu^{(a)} u|^2 + |D_\mu^{(b)} v|^2 + \tag{89} \\ & + i(c_{\bar{z}} \bar{v} \circ \partial_z u - c_z \partial_{\bar{z}} \bar{u} \circ v + c_z \bar{u} \circ \partial_z v - c_{\bar{z}} \partial_z \bar{v} \circ u) + c_z c_{\bar{z}} (|u|^2 + |v|^2) + \\ & + i\lambda_1 (\|u\|^2 - N) + i\lambda_2 (\|v\|^2 - N) + i\tau \bar{u} \circ v + i\bar{\tau} \bar{v} \circ u + \text{terms with ghost fields.} \end{aligned}$$

The two $\mathbb{C}\mathbb{P}^{N-1}$ models, parametrized by vectors u and v , interact via the matter field c , as well as through the orthogonality constraint $\bar{u} \circ v = 0$, implemented with the help of the Lagrange multiplier τ . The first line in (89) encodes the propagators and interaction vertices of two $\mathbb{C}\mathbb{P}^{N-1}$ models – these are shown in Fig. 4 and Fig. 5 respectively. Apart from these, our model has the additional complex fields τ and c_z transforming in the bi-fundamental representation of $U(1) \times U(1)$. Their respective propagators and interactions are shown in Fig. 6.

In implementing the steepest descent method for the large- N limit, we will parametrize $\Lambda = \Lambda_0 + \frac{1}{\sqrt{N}} \tilde{\Lambda}$, $\mathcal{A} = \frac{1}{\sqrt{N}} \tilde{\mathcal{A}}$. The expansion of the effective action \mathcal{S}^{eff} starts at the quadratic order in $\tilde{\lambda}, \tilde{\mathcal{A}}$, and all subsequent orders are suppressed by powers of \sqrt{N} . The propagators shown in the figures are obtained by expanding the effective action in (86) to quadratic order in the fields, which yields in momentum space:

$$\begin{aligned} \mathcal{S}^{\text{eff}} = & \int d^2p \left(\Gamma_{\mu\nu}(\vec{p}) \times [a_\mu(\vec{p}) a_\nu(-\vec{p}) + b_\mu(\vec{p}) b_\nu(-\vec{p})] + \tag{90} \right. \\ & \left. + \tilde{\Gamma}(p) c_{\bar{z}}(\vec{p}) c_z(-\vec{p}) + \Gamma_\lambda(\vec{p}) \text{Tr}(\lambda(p)\lambda(-p)) \right) \end{aligned}$$

The functions $\Gamma_{\mu\nu}(\vec{p})$ and $\Gamma_\lambda(\vec{p})$ were computed already in [10]. The function $\tilde{\Gamma}(p)$ is found in a similar way, and altogether we have:

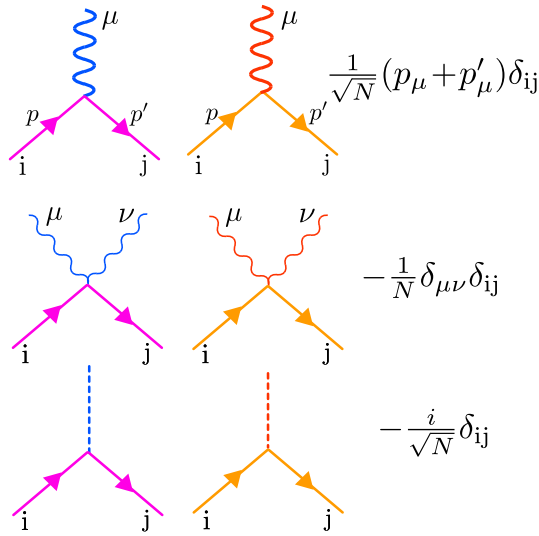


Fig. 5. The vertices of the (doubled) CP^{N-1} model.

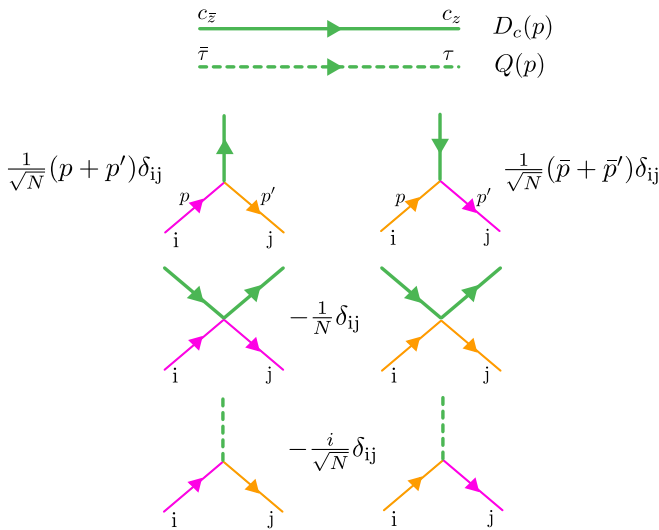


Fig. 6. Propagators and vertices of the new fields τ, c_z . Here $p = p_z$ and $\bar{p} = p_{\bar{z}}$ are the holomorphic/anti-holomorphic components of the momentum.

$$\Gamma_\lambda(\vec{p}) = \frac{1}{2} A(p) = \frac{1}{2} \int \frac{d^2k}{(k^2 + m^2)((k + p)^2 + m^2)} \tag{91}$$

$$\Gamma_{\mu\nu}(\vec{p}) = \frac{1}{2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left((p^2 + 4m^2) A(p) - \frac{1}{\pi} \right) \tag{92}$$

$$\tilde{\Gamma}(p) = 2\Gamma_{z\bar{z}}(\vec{p}) = \frac{1}{2} \left((p^2 + 4m^2) A(p) - \frac{1}{\pi} \right) \tag{93}$$

In the effective action there are no mixed terms of the form $\int d^2 p \Gamma_\mu(p) a_\mu(p) \lambda(p)$ due to the fact that the corresponding one-loop diagram vanishes:

$$\Gamma_\mu(p) = \int \frac{d^2 k}{(2\pi)^2} \frac{2k_\mu + p_\mu}{(k^2 + m^2)((k + p)^2 + m^2)} = \{k \rightarrow -p - k\} = -\Gamma_\mu(p) = 0. \tag{94}$$

In the Lorenz gauge the propagators featuring in the figures have the following expressions:

$$G(p) = \frac{1}{p^2 + m^2}, \quad Q(p) = A(p)^{-1} \tag{95}$$

$$D_{\mu\nu}(\vec{p}) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left[(p^2 + 4m^2) A(p) - \frac{1}{\pi} \right]^{-1} \tag{96}$$

$$D_c(p) = 2 \left[(p^2 + 4m^2) A(p) - \frac{1}{\pi} \right]^{-1} \tag{97}$$

6. The Wilson loop and conserved charges

The main object of interest that we wish to be able to calculate is the following correlation function, with a Wilson loop of the flat connection inserted:

$$Q(\Gamma) := \left(\prod_{i \in I} u_i(z_i, \bar{z}_i) \prod_{j \in J} v_j(z_j, \bar{z}_j) \prod_{k \in K} \mathcal{A}_k(z_k, \bar{z}_k) \right) P_0 e^{-\int_\Gamma \mathcal{A}_u}, \tag{98}$$

where \mathcal{A}_u is given by formula (34). The notation P_0 is meant to emphasize that the Wilson loop depends on the choice of a base point on the contour, which we call $p_0 \in \Gamma$. On the worldsheet $\Sigma \simeq \mathbb{R}^2$, when the contour Γ is infinite, it is often convenient to choose infinity as the base point. The Noether current K entering this formula is the one corresponding to the $U(N)$ -symmetry acting on the φ fields: $\varphi \rightarrow g \circ \varphi$, $g \in U(N)$. Therefore the Noether current one-form is (see (70))

$$K = \frac{2g^2}{N} (\varphi (\mathcal{D}_z \varphi)^\dagger d\bar{z} - \mathcal{D}_z \varphi \varphi^\dagger dz). \tag{99}$$

The notation $Q(\Gamma)$ is meant to emphasize that we will be interested in the dependence on the contour Γ . Note that classically the connection \mathcal{A}_u is flat, hence $P_0 e^{-\int_\Gamma \mathcal{A}_u}$ is independent of Γ , provided one fixes the homology class of Γ (and assuming the base point p_0 is left intact). Nevertheless, the quantity $Q(\Gamma)$ will in general depend on Γ due to the UV-divergences that will need to be regularized,⁴ and this gives rise to the quantum anomalies in the higher conservation laws of the model, as first calculated in [17].

6.0.1. Example

It is useful to consider a correlation function with an insertion of a conserved charge \mathcal{Q} in a simpler model. In fact, the model will be as simple as possible, namely a two-dimensional theory with the Lagrangian

$$\mathcal{L} = \partial_z \phi \partial_{\bar{z}} \phi. \tag{100}$$

⁴ Renormalization of Wilson loops in gauge theories, which also exhibit UV-divergences, was discussed as early as in [55].

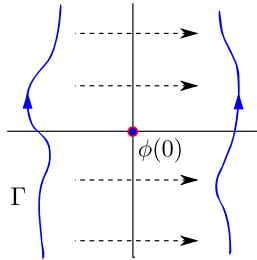


Fig. 7. Moving the contour Γ past an operator inserted at the origin.

This Lagrangian exhibits a shift symmetry of the form $\phi \rightarrow \phi + a$, where a is a constant. The corresponding Noether current is $J = \partial_z \phi dz + \partial_{\bar{z}} \phi d\bar{z}$ and the conserved charge is

$$Q = \int_{\Gamma} *J = \int_{\Gamma} i (\partial_z \phi dz - \partial_{\bar{z}} \phi d\bar{z}). \tag{101}$$

The only connected correlation function to consider is $\langle Q \phi(0) \rangle$, where we have chosen to place the second operator at the origin. The contour Γ will run along the y -axis at a fixed value of x , though one could as well consider a curved contour, as shown in Fig. 7. Using the fact that the two-dimensional propagator is $\langle \phi(z, \bar{z}) \phi(0) \rangle = -\frac{1}{4\pi} \log |z|^2$, we get for the correlation function

$$\langle Q \phi(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x dy}{x^2 + y^2} = \frac{1}{2} \text{sgn}(x). \tag{102}$$

The value is independent of the position of the contour Γ , as long as it does not cross the point of insertion of the local operator (in this case the origin). The discontinuity of the correlation function that arises when the contour is moved past a local operator is compatible with the Ward identity for the symmetry in question. ■

The first non-local charge (as well as the usual local charge) may be calculated by expanding the path-ordered exponent $\mathcal{W}_0(u) := P_0 e^{-\int_{\Gamma} \mathcal{A}_u}$ around the point $u = 1$ to second order. In order to obtain charges that are manifestly (anti)-Hermitian, it will be useful to parametrize $u = e^{-i\theta}$ and expand to second order in θ . First we note that, to this order,

$$\mathcal{A}_u = \frac{i\theta}{2} (K_z dz - K_{\bar{z}} d\bar{z}) + \frac{\theta^2}{4} (K_z dz + K_{\bar{z}} d\bar{z}) + \dots = \tag{103}$$

$$= \frac{\theta}{2} *K + \frac{\theta^2}{4} K + \dots \tag{104}$$

Here $*$ is the Hodge star, defined by $*dz = i d\bar{z}$. We are now in a position to expand the path-ordered exponent:

$$\mathcal{W}_0(u) = \mathbb{1} - \frac{\theta}{2} \int_{\Gamma} *K - \frac{\theta^2}{4} \left[\int_{\Gamma} K - P \int_{\Gamma} *K \int_{\Gamma} *K \right] + \dots \tag{105}$$

The two first conserved charges are therefore

$$Q_1 = \int_{\Gamma} *K \tag{106}$$

$$\widehat{Q}_2 = \int_{\Gamma} K - P \int_{\Gamma} *K \int_{\Gamma} *K . \tag{107}$$

Q_1 is the usual Noether charge corresponding to the $SU(N)$ -symmetry. The ‘conservation’ of the charges means in this context that they are independent of smooth deformations of the contour Γ . For the charge Q_1 this follows from the Stokes theorem and the conservation of the current, i.e. $d * K = 0$. We will follow the conventions of [17] and further simplify the charge \widehat{Q}_2 , using the fact that we are free to add to it an arbitrary function of the charge Q_1 . The square of this charge is

$$Q_1^2 = \int dt ds (*K)_t (*K)_s = \int_{t < s} + \int_{t > s} = \int_{t < s} dt ds [(*K)_t (*K)_s + (*K)_s (*K)_t]$$

Therefore we introduce the charge

$$Q_2 = \widehat{Q}_2 + \frac{1}{2} Q_1^2 = \int_{\Gamma} K - \frac{1}{2} \int_{t < s} dt ds [(*K)_t, (*K)_s] \tag{108}$$

The charge so defined also has the nice property that it lies in the Lie algebra of G , $Q_2 \in \mathfrak{g}$. In order to automatically obtain charges with values in the Lie algebra, one should consider the ‘Berry connection’ $\mathcal{B} = (\mathcal{W}_0(u))^{-1} \frac{d\mathcal{W}_0(u)}{d\theta}$ and expand it around $\theta = 0$.

Let us check that, classically, the charge Q_2 is independent of Γ . To this end we introduce the one-form (the lower limit of integration – point p_0 – is again the base point, which enters the definition of the Wilson loop)

$$S(p) := \left[\left(\int_{p_0}^p *K \right), *K(p) \right]. \tag{109}$$

Here $\int_{p_0}^p *K$ is the primitive of the one-form $*K$, which is well-defined due to the fact that $*K$ is closed (conservation of the current). In fact, if the point p was always restricted to lie on Γ , the integral $\int_{p_0}^p *K$ could be understood as an integral along Γ , even without the closedness of $*K$.

However we will now wish to appeal again to Stokes’ theorem in the interior of $D_{\delta\Gamma}$ (this is the space between the two contours, see Fig. 11), and therefore we will need a one-form well-defined inside $D_{\delta\Gamma}$.

With this said, we may now write the charge Q_2 as

$$Q_2 = \int_{\Gamma} \left(K - \frac{1}{2} S \right). \tag{110}$$

The variation again may be computed using Stokes’ theorem. Taking into account that $dS = 2 * K \wedge *K = 2 K \wedge K$ and that the current K is flat, $dK - K \wedge K = 0$, we find that the variation is zero:

$$\delta_{\Gamma} Q_2(\Gamma) = 0. \tag{111}$$

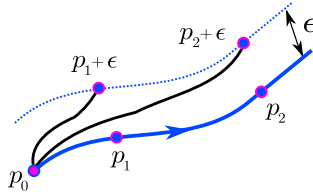


Fig. 8. The paths of integration from the base point p_0 to a given point $p + \epsilon$ in the definition of the operator S_ϵ . The points p_0, p_1, p_2 lay on the contour Γ .

6.1. The regularized charge

In the quantum theory, due to the short-distance singularities in the product of two currents, one needs to define a regularized version of the non-local charge Q_2 . This will involve the splitting of coincident points in the definition of the one-form S in (109). One possibility is to define the following ϵ -regularized form:

$$S_\epsilon(p) := \left[\left(\int_{p_0}^{p+\epsilon} *K \right), *K(p) \right]. \tag{112}$$

Here it is implied that ϵ is a fixed two-vector, and we are considering the case when $\Sigma_2 \simeq \mathbb{R}^2$, equipped with a flat metric, when the notion of addition ‘ $p + \epsilon$ ’ makes sense. The one-form S_ϵ depends on the contour of integration in (112) in the topological sense, i.e. it depends on whether the contour winds around the point p . Therefore we will fix a certain value of ϵ (such as the one shown in Fig. 8) and define the auxiliary contours in (112) for this value of ϵ . For instance, for contours Γ of the type shown in Fig. 8 one can define the auxiliary contour by requiring that it lay on one side of Γ . Then we can gradually change the value of ϵ so that it circles around the origin, $\epsilon \rightarrow e^{2\pi i} \epsilon$. As a result, S_ϵ will gain a monodromy of the form

$$S_{e^{2\pi i} \epsilon} - S_\epsilon = \left[\left(\oint *K \right), *K(p) \right], \tag{113}$$

where the integral is around a loop centered at p . Since the current K is conserved, $d * K = 0$, the value of the integral does not depend on the size of the loop. Yet it is not zero, due to the presence of the operator insertion of $*K(p)$ at the center of the loop. However, the loop may be shrunk to be arbitrarily small, and the value of the commutator may be calculated from the most singular term in the OPE of the two currents. Such analysis below will bring us to the following answer:

$$\left[\left(\oint *K \right), *K(p) \right] = 2g^2 *K(p) \tag{114}$$

In order to cancel this monodromy, we will consider the operator

$$Q_\epsilon(\Gamma) := \int_\Gamma \left(\left[\mathbf{a} + \frac{1}{2\pi} \log(\epsilon) \right] K_z dz + \left[\bar{\mathbf{a}} + \frac{1}{2\pi} \log(\bar{\epsilon}) \right] K_{\bar{z}} d\bar{z} - \frac{1}{2g^2} S_\epsilon \right), \tag{115}$$

where \mathbf{a} is a constant, independent of ϵ , to be chosen later. It is clear that the operator defined above is invariant as the regularization parameter ϵ rotates around the origin: $\epsilon \rightarrow e^{2\pi i} \epsilon$. This operator Q_ϵ is our candidate for the regularized version of the charge Q_2 .

We wish to prove the following:

- There exists a limit $\lim_{\epsilon \rightarrow 0} Q_\epsilon$
- The limit depends on the curve Γ through an anomaly 2-form Ω_A , namely

$$\delta_\Gamma \left(\lim_{\epsilon \rightarrow 0} Q_\epsilon \right) = \int_{D_{\delta\Gamma}} \Omega_A, \tag{116}$$

where $D_{\delta\Gamma}$ is a two-dimensional domain bounded by the original and final curves Γ_1, Γ_2 (see Fig. 11).

Let us start from the second point by calculating the difference $Q_\epsilon(\Gamma_1) - Q_\epsilon(\Gamma_2)$ of the values of the charge on two different contours. We use Stokes’ theorem and the conservation of the current to obtain:

$$Q_\epsilon(\Gamma_1) - Q_\epsilon(\Gamma_2) = \int_{D_{\delta\Gamma}} dz \wedge d\bar{z} \left[\left(\text{Re}(\mathbf{a}) + \frac{1}{2\pi} \log |\epsilon| \right) (\partial_z K_{\bar{z}} - \partial_{\bar{z}} K_z) - \frac{1}{2g^2} ([K_z(p + \epsilon), K_{\bar{z}}(p)] + [K_z(p), K_{\bar{z}}(p + \epsilon)]) \right]. \tag{117}$$

Therefore one of the goals will be the calculation of the OPE of the two currents $[K_z(p + \epsilon), K_{\bar{z}}(p)]$ as $\epsilon \rightarrow 0$.

6.2. The operator product expansion

We start by recalling the definition of the current (70):

$$K = \frac{2g^2}{N} \left(\varphi (\mathcal{D}_z \varphi)^\dagger d\bar{z} - \mathcal{D}_z \varphi \varphi^\dagger dz \right). \tag{118}$$

First we will calculate the OPE between two holomorphic components of the current, and as a result we will be able to prove the first point above – that there exists a limit $\lim_{\epsilon \rightarrow 0} Q_\epsilon$.

In what follows we will encounter the function $D(z)$ – the massive scalar propagator in two dimensions (recalling the definition $\{z, \bar{z}\} = \frac{x^1 \pm ix^2}{\sqrt{2}}$ of the complex coordinates):

$$D(z) = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{i(p_1 x^1 + p_2 x^2)}}{p_1^2 + p_2^2 + m^2} = \frac{1}{2\pi} K_0(\sqrt{2}m|z|) = -\frac{1}{4\pi} \left(1 + \frac{m^2|z|^2}{2} \right) \log \left(\frac{m^2|z|^2}{2} \right) - \frac{\gamma}{2\pi} + \frac{1-\gamma}{2\pi} \frac{m^2|z|^2}{2} + \dots \tag{119}$$

The final line captures the first few terms in the expansion of $D(z)$ for $z \rightarrow 0$.

Let us also make a reservation regarding the technique that we use in the present section. Most of the diagrams that we will be calculating (see Figs. 12 and 13) really correspond to the theory of a scalar field φ in a background gauge field \mathcal{A} . Therefore these diagrams are the same as the ones that would arise in the corresponding Grassmannian sigma-model with target space $G_{M,N}$. For this reason in these calculations we will simply treat \mathcal{A} as a non-abelian external gauge field (which can later be restricted to the form (80)). To simplify the figures, we will still be drawing

the same diagrams as for a single copy of the CP^{N-1} -model, but with the understanding that the fields carry an additional index. The sole role of this index is to make sure the ordering of the fields is taken into account. The only diagrams where the gauge field enters in the internal lines are the ones of Fig. 14, and for these diagrams we carry out a more thorough analysis in Section 6.2.4.

6.2.1. The commutator $[(\ast K)_z(z_1), (\ast K)_z(z_2)]$

We now pass to the calculation of the OPE of the commutator of two $(\ast K)_z$ -components. To this end we consider the product

$$(\ast K)_z(z_1) \cdot (\ast K)_z(z_2) = -\frac{4g^4}{N^2} \mathcal{D}_z \varphi(z_1) \left(\varphi^\dagger(z_1) \mathcal{D}_z \varphi(z_2) \right) \varphi^\dagger(z_2) \tag{120}$$

We write

$$\varphi^\dagger(z_1) \mathcal{D}_z \varphi(z_2) = -N \cdot \partial_{z_1} D(z_1 - z_2) \mathbb{1}_2 + O(\log(|z_1 - z_2|)) \tag{121}$$

Note that in the commutator $[(\ast K)_z(z_1), (\ast K)_z(z_2)]$ the logarithmic (as well as finite) terms will disappear as they are symmetric under $z_1 \leftrightarrow z_2$, therefore we have:

$$[(\ast K)_z(z_1), (\ast K)_z(z_2)] = \frac{8g^4}{N} \partial_{z_1} D(z_1 - z_2) : \mathcal{D}_z \varphi(z_1) \varphi^\dagger(z_2) : + \dots \tag{122}$$

We have inserted a normal ordered product in the r.h.s., since the self-contractions of the field φ would give a contribution proportional to the unit operator $\mathbb{1}_N$, which clearly cannot enter in the expansion of a commutator.

From (119) it follows that $\partial_{z_1} D(z_1 - z_2) = -\frac{1}{4\pi} \frac{1}{z_1 - z_2} + O(z_1 - z_2)$, where $O(z_1 - z_2)$ denotes terms that vanish in the limit $z_1 \rightarrow z_2$. Therefore

$$[(\ast K)_z(z_1), (\ast K)_z(z_2)] = \frac{g^2}{\pi} \frac{1}{z_1 - z_2} K_z(z_2) + O(|z_1 - z_2|^0) \tag{123}$$

Analogously for the \bar{z} -components we have:

$$[(\ast K)_{\bar{z}}(z_1), (\ast K)_{\bar{z}}(z_2)] = \frac{g^2}{\pi} \frac{1}{\bar{z}_1 - \bar{z}_2} K_{\bar{z}}(z_2) + O(|z_1 - z_2|^0) \tag{124}$$

The above two equalities are already sufficient to prove (114). Indeed, in the limit of an infinitesimally small contour around w we have

$$\begin{aligned} & \left[\left(\oint (\ast K)_z dz + \oint (\ast K)_{\bar{z}} d\bar{z} \right), \ast K(w) \right] = \\ &= \oint \frac{g^2}{\pi} \frac{dz}{z - w} K_w(w) dw + \oint \frac{g^2}{\pi} \frac{d\bar{z}}{\bar{z} - \bar{w}} K_{\bar{w}}(w) d\bar{w} = \\ &= 2g^2 (i K_w(w) dw - i K_{\bar{w}}(w) d\bar{w}) = 2g^2 \ast K(w) \end{aligned} \tag{125}$$

A similar derivation is used to find the behavior of S_ϵ for $\epsilon \rightarrow 0$:

$$\begin{aligned} S_\epsilon(w) &= \left[\left(\int_0^{w+\epsilon} (\ast K)_z dz + \int_0^{\bar{w}+\bar{\epsilon}} (\ast K)_{\bar{z}} d\bar{z} \right), \ast K(w) \right] \sim \\ &\underset{\epsilon \rightarrow 0}{\sim} \frac{g^2}{\pi} \log(\epsilon) K_w(w) dw + \frac{g^2}{\pi} \log(\bar{\epsilon}) K_{\bar{w}}(w) d\bar{w} + \dots, \end{aligned} \tag{126}$$

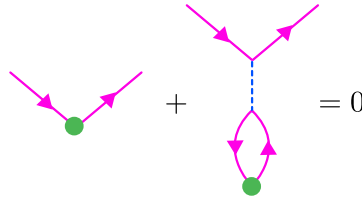


Fig. 9. A proof that $\langle \varphi^\dagger(z)\varphi(z) \rangle := 0$.

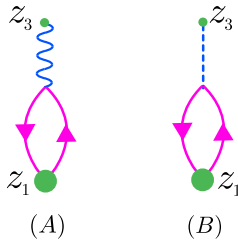


Fig. 10. Diagrams that contribute to the correlation functions (A) $\langle \varphi^\dagger(z_1)\varphi(z_2)\mathcal{A}(z_3) \rangle$ and (B) $\langle \varphi^\dagger(z_1)\varphi(z_2)\lambda(z_3) \rangle$ in the limit $z_2 \rightarrow z_1$.

where ... denotes finite terms. It is now obvious that the integrand in (115) has a finite limit for $\epsilon \rightarrow 0$. This limit is, in particular, independent of the angle at which ϵ approaches zero.

6.2.2. The commutator $[(\ast K)_z(z_1), (\ast K)_{\bar{z}}(z_2)]$

First of all, using the above definition, we calculate

$$\ast K = -\frac{2ig^2}{N} \left(\mathcal{D}_z \varphi \varphi^\dagger dz + \varphi (\mathcal{D}_{\bar{z}} \varphi)^\dagger d\bar{z} \right), \tag{127}$$

therefore

$$(\ast K)_z(z_1) \cdot (\ast K)_{\bar{z}}(z_2) = -\frac{4g^4}{N^2} \mathcal{D}_z \varphi(z_1) \left(\varphi^\dagger(z_1)\varphi(z_2) \right) (\mathcal{D}_{\bar{z}} \varphi)^\dagger(z_2) \tag{128}$$

Classically $\varphi^\dagger(z_1)\varphi(z_1) = \frac{N}{g^2} \mathbb{1}_2$. Here however we need to write down the OPE of the following form (such an OPE was first considered in [56])

$$\varphi^\dagger(z_1)\varphi(z_2) = N \cdot D(z_1 - z_2) \mathbb{1}_2 + \langle \varphi^\dagger(z_1)\varphi(z_1) \rangle + \dots \tag{129}$$

where ... denotes terms that vanish as $z_2 \rightarrow z_1$. We will now show that $\langle \varphi^\dagger(z_1)\varphi(z_1) \rangle := 0$. Indeed, this follows from the cancellation of two diagrams shown in Fig. 9.

Note that, in principle, one should also consider the correlation functions $\langle \varphi^\dagger(z_1)\varphi(z_2)\mathcal{A}(z_3) \rangle$ and $\langle \varphi^\dagger(z_1)\varphi(z_2)\lambda(z_3) \rangle$ (see Fig. 10). One can show, however, that

$$\lim_{z_2 \rightarrow z_1} \langle \varphi^\dagger(z_1)\varphi(z_2)\mathcal{A}(z_3) \rangle = 0, \tag{130}$$

$$\lim_{z_2 \rightarrow z_1} \langle \varphi^\dagger(z_1)\varphi(z_2)\lambda(z_3) \rangle \sim \delta(z_1 - z_3). \tag{131}$$

The diagram (A) vanishes for symmetry reasons, as its value is equal to Γ_μ from (94). As for the second diagram, the loop integral is precisely equal to the inverse of the λ -field propagator

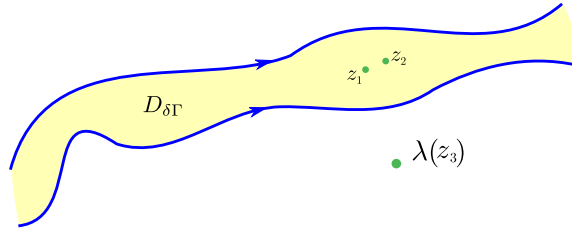


Fig. 11. The point z_3 , where the operator λ is inserted, is outside of the relevant integration domain $D_{\delta\Gamma}$. Since according to (131) the corresponding correlation function (132) is proportional to $\delta(z_1 - z_3)$, it is zero for the depicted configuration.

$Q(p)$, and therefore in coordinate space the diagram (B) is proportional to $\delta(z_1 - z_3)$. Such a term does not contribute to the anomaly, as at the level of correlation functions we have the following result (see Fig. 11):

$$\langle \delta_\Gamma Q \cdot \lambda(z_3) \rangle = \int_{D_{\delta\Gamma}} \langle \Omega_A \cdot \lambda(z_3) \rangle = 0 \quad \text{for } z_3 \text{ outside of } D_{\delta\Gamma}. \tag{132}$$

Let us also recall that $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^N$ is a $(N \times 2)$ -matrix. Let us denote the i -th row by φ^i and, following [17], consider the off-diagonal entries of $((*K)_z(z_1) \cdot (*K)_{\bar{z}}(z_2))^{ij}$ ($i \neq j$) in the OPE. This will slightly simplify things, as in this case the contractions between the outer fields φ^i and $(\varphi^\dagger)^j$ are prohibited. Then we get

$$((*K)_z(z_1) \cdot (*K)_{\bar{z}}(z_2))^{ij} \xrightarrow{z_2 \rightarrow z_1} -\frac{4g^4}{N} \left[\mathcal{D}_z \varphi \mathcal{D}_{\bar{z}} \varphi^\dagger \right]_{z_1}^{ij} D(z_1 - z_2), \tag{133}$$

where it is understood that one should only keep the non-vanishing terms in the expansion of $D(z_1 - z_2)$ as $z_2 \rightarrow z_1$. Using the definition of the current K , we find that

$$\begin{aligned} \partial_z K_{\bar{z}} - \partial_{\bar{z}} K_z &= \frac{4g^2}{N} \mathcal{D}_z \varphi \mathcal{D}_{\bar{z}} \varphi^\dagger + \frac{2g^2}{N} (\varphi \mathcal{D}_z \mathcal{D}_{\bar{z}} \varphi^\dagger + \mathcal{D}_{\bar{z}} \mathcal{D}_z \varphi \varphi^\dagger) = \\ &= \frac{4g^2}{N} \mathcal{D}_z \varphi \mathcal{D}_{\bar{z}} \varphi^\dagger + \frac{2g^2}{N} \varphi (i\lambda + iF_{z\bar{z}}) \varphi^\dagger, \end{aligned} \tag{134}$$

where in the last line we have used the e.o.m. (72). Substituting this in the above OPE, we get

$$\begin{aligned} ((*K)_z(z_1) \cdot (*K)_{\bar{z}}(z_2))^{ij} &\xrightarrow{z_2 \rightarrow z_1} \\ &\rightarrow g^2 \left(\partial_{\bar{z}} K_z - \partial_z K_{\bar{z}} + \frac{2g^2}{N} \left[\varphi (i\lambda + iF_{z\bar{z}}) \varphi^\dagger \right] \right)_{z_1}^{ij} D(z_1 - z_2) \end{aligned} \tag{135}$$

Multiplying the currents in the opposite order, we obtain

$$(*K)_{\bar{z}}(z_1) \cdot (*K)_z(z_2) = -\frac{4g^4}{N^2} \varphi(z_1) \left(\mathcal{D}_{\bar{z}} \varphi^\dagger(z_1) \cdot \mathcal{D}_z \varphi(z_2) \right) \varphi^\dagger(z_2) \tag{136}$$

Classically we see from (75) that we could replace $\lim_{z_2 \rightarrow z_1} \frac{g^2}{N} \mathcal{D}_{\bar{z}} \varphi^\dagger(z_1) \cdot \mathcal{D}_z \varphi(z_2) = -\frac{1}{2} i\lambda(z_1) - \frac{1}{2} i F_{z\bar{z}}(z_1)$. At the quantum level the corresponding statement is that the following OPE holds:

$$\frac{1}{N} \mathcal{D}_{\bar{z}} \varphi^\dagger(z_1) \cdot \mathcal{D}_z \varphi(z_2) \xrightarrow{z_2 \rightarrow z_1} \kappa_\lambda(z_1 - z_2) i\lambda(z_1) + \kappa_F(z_1 - z_2) i F_{z\bar{z}}(z_1), \tag{137}$$

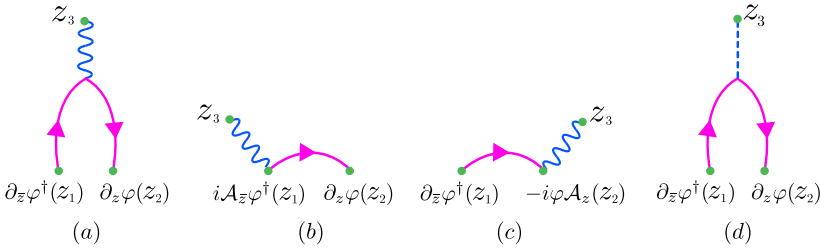


Fig. 12. The diagrams contributing at order $\frac{1}{\sqrt{N}}$ to the OPE (137).

where $\kappa_\lambda(z)$ and $\kappa_F(z)$ are two coefficient functions to be found, at leading order $\frac{1}{\sqrt{N}}$, from the diagrams shown in Fig. 12.

The sum of diagrams (a), (b) and (c) leads to the following integral^{5,6}:

$$I = \int \frac{d^2k}{(2\pi)^2} \left(\frac{k_{\bar{z}}(k_z + p_z)}{(k^2 + m^2)((k + p)^2 + m^2)} ((2k_{\bar{z}} + p_{\bar{z}})\mathcal{A}_z(p) + (2k_z + p_z)\mathcal{A}_{\bar{z}}(p)) - \frac{k_{\bar{z}}}{k^2 + m^2} \mathcal{A}_z(p) - \frac{k_z + p_z}{(p + k)^2 + m^2} \mathcal{A}_{\bar{z}}(p) \right) e^{ik\epsilon} \tag{138}$$

By a direct calculation one can show (see Appendix D) that, up to terms which vanish in the limit $\epsilon \rightarrow 0$, it is equal to

$$I = (p_{\bar{z}}\mathcal{A}_z - p_z\mathcal{A}_{\bar{z}}) (\Gamma_{z\bar{z}} + \frac{1}{4\pi}) - \frac{1}{2}(p_{\bar{z}}\mathcal{A}_z - p_z\mathcal{A}_{\bar{z}}) D(\epsilon) + \dots, \tag{139}$$

where $\Gamma_{z\bar{z}}$ is a component of the $\Gamma_{\mu\nu}$ -tensor defined in the effective action (90). The contribution of $\Gamma_{z\bar{z}}(p)$ may be discarded in the OPE, as again it leads to contact terms in the correlation functions (analogously to the situation described in Fig. 11).

Let us now evaluate the diagram (d). It corresponds to the integral

$$I_d = -i\lambda(p) \int \frac{d^2k}{(2\pi)^2} \frac{k_{\bar{z}}(k_z + p_z)}{(k^2 + m^2)((k + p)^2 + m^2)} \times e^{ik\epsilon} \tag{140}$$

Using manipulations similar to the ones described in Appendix D, one can show that up to terms vanishing in the limit $\epsilon \rightarrow 0$ this integral is equal to

$$I_d = (m^2 + p_z p_{\bar{z}}) \Gamma_\lambda(p) i\lambda(p) - \frac{1}{2}i\lambda(p) D(\epsilon) + \dots \tag{141}$$

Once again, the terms proportional to $\Gamma_\lambda(p) i\lambda(p)$ are contact terms and may be dropped in the OPE.

⁵ Note that here, and in what follows, we use the non-rescaled fields \mathcal{A}, λ in the external lines of various diagrams. As explained in Section 5, to justify the result within the $\frac{1}{\sqrt{N}}$ -expansion, one should keep in mind that $\lambda = \frac{1}{\sqrt{N}}\tilde{\lambda}, \mathcal{A} = \frac{1}{\sqrt{N}}\tilde{\mathcal{A}}$.

It is in this sense that we count the orders of the $\frac{1}{\sqrt{N}}$ -expansion in Figs. 12, 13.

⁶ In writing out explicit expressions for the Feynman integrals we will suppress factors of N , coming from loops of matter fields, and reinstate them only in the final expressions for the OPE's.

rightmost propagator in Fig. (a'). Cancelling the propagator $\frac{1}{k^2+m^2}$, for example, one is left with the following integral (after the change of variable $k \rightarrow k - p_1$):

$$I_{a'}^{(1)} \simeq \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} e^{ik\epsilon} \frac{\mathcal{A}_\mu(p_1)(2k - p_1)_\mu (2k + p_2)_\nu \mathcal{A}_\nu(p_2)}{((k + p_2)^2 + m^2)(k^2 + m^2)} + (p_1 \leftrightarrow p_2) \tag{145}$$

Here p_1 and p_2 are the momenta carried by the external gauge fields, and \simeq means ‘up to terms that vanish in the limit $\epsilon \rightarrow 0$ ’. It is easily seen that in the bracket $(2k - p_1)_\mu$ one can drop the term, proportional to p_1 , as the integral multiplying this term vanishes by symmetry in the limit $\epsilon \rightarrow 0$. In this limit the integral is indistinguishable from

$$\begin{aligned} I_{a'}^{(1)} &\simeq \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} e^{ik\epsilon} \frac{\mathcal{A}_\mu(p_1)(2k + p_2)_\mu (2k + p_2)_\nu \mathcal{A}_\nu(p_2)}{((k + p_2)^2 + m^2)(k^2 + m^2)} + (p_1 \leftrightarrow p_2) = \\ &= \frac{1}{4} \tau_{z\bar{z}}(p_2|\epsilon) (\mathcal{A}_z(p_1)\mathcal{A}_{\bar{z}}(p_2) + \mathcal{A}_{\bar{z}}(p_1)\mathcal{A}_z(p_2)) + \\ &\quad + \frac{1}{4} \tau_{\bar{z}\bar{z}}(p_2|\epsilon) \mathcal{A}_z(p_1)\mathcal{A}_z(p_2) + \frac{1}{4} \tau_{zz}(p_2|\epsilon) \mathcal{A}_{\bar{z}}(p_1)\mathcal{A}_{\bar{z}}(p_2) + (p_1 \leftrightarrow p_2), \end{aligned} \tag{146}$$

where we have introduced the notation

$$\tau_{\mu\nu}(p|\epsilon) := \int \frac{d^2k}{(2\pi)^2} e^{ik\epsilon} \frac{(2k + p)_\mu (2k + p)_\nu}{((k + p)^2 + m^2)(k^2 + m^2)}. \tag{147}$$

Apart from the contribution $I_{a'}^{(1)}$, there will be a contribution $I_{a'}^{(2)}$, where instead of cancelling the propagator $\frac{1}{k^2+m^2}$ one cancels the propagator $\frac{1}{(k+p)^2+m^2}$. The only difference in the resulting expression will be that in the argument of the function τ one will have to replace $p_1 \rightarrow p_2$.

We now pass to the calculation of diagrams (b') and (c'). Their sum is

$$\begin{aligned} I_{b'} + I_{c'} &\simeq \\ &\simeq - \int \frac{d^2k}{(2\pi)^2} e^{ik\epsilon} \frac{\mathcal{A}_{\bar{z}}(p_1)k_z (2k + p_2)_\nu \mathcal{A}_\nu(p_2) + (2k + p_2)_\nu \mathcal{A}_\nu(p_2)\mathcal{A}_z(p_1)k_{\bar{z}}}{((k + p_2)^2 + m^2)(k^2 + m^2)} + \\ &+ (p_1 \leftrightarrow p_2) \simeq -\frac{1}{2} \tau_{z\bar{z}}(p_2|\epsilon) (\mathcal{A}_{\bar{z}}(p_1)\mathcal{A}_z(p_2) + \mathcal{A}_{\bar{z}}(p_2)\mathcal{A}_z(p_1)) - \\ &- \frac{1}{2} \tau_{\bar{z}\bar{z}}(p_2|\epsilon) \mathcal{A}_z(p_2)\mathcal{A}_z(p_1) - \frac{1}{2} \tau_{zz}(p_2|\epsilon) \mathcal{A}_{\bar{z}}(p_1)\mathcal{A}_{\bar{z}}(p_2) + (p_1 \leftrightarrow p_2) \end{aligned} \tag{148}$$

Altogether (146), a similar expression with the replacement $\tau(p_2) \rightarrow \tau(p_1)$ (which we call $I_{a'}^{(2)}$) and (148) give the following result:

$$\begin{aligned} I_{a'}^{(1)} + I_{a'}^{(2)} + I_{b'} + I_{c'} &= \\ &= \frac{1}{4} \tau_{z\bar{z}}(p_2|\epsilon) ([\mathcal{A}_z(p_1), \mathcal{A}_{\bar{z}}(p_2)] + [\mathcal{A}_z(p_2), \mathcal{A}_{\bar{z}}(p_1)]) + \\ &+ \frac{1}{4} \tau_{\bar{z}\bar{z}}(p_2|\epsilon) [\mathcal{A}_z(p_1), \mathcal{A}_z(p_2)] + \frac{1}{4} \tau_{zz}(p_2|\epsilon) [\mathcal{A}_{\bar{z}}(p_2), \mathcal{A}_{\bar{z}}(p_1)] + (p_1 \leftrightarrow p_2) = \\ &= \frac{1}{4} (\tau_{z\bar{z}}(p_2|\epsilon) + \tau_{z\bar{z}}(p_1|\epsilon)) ([\mathcal{A}_z(p_1), \mathcal{A}_{\bar{z}}(p_2)] + [\mathcal{A}_z(p_2), \mathcal{A}_{\bar{z}}(p_1)]) + \\ &+ \frac{1}{4} (\tau_{\bar{z}\bar{z}}(p_2|\epsilon) - \tau_{\bar{z}\bar{z}}(p_1|\epsilon)) [\mathcal{A}_z(p_1), \mathcal{A}_z(p_2)] + \\ &+ \frac{1}{4} (\tau_{zz}(p_2|\epsilon) - \tau_{zz}(p_1|\epsilon)) [\mathcal{A}_{\bar{z}}(p_2), \mathcal{A}_{\bar{z}}(p_1)] \end{aligned} \tag{149}$$

We will now relate the functions $\tau_{\mu\nu}$ to the effective action $\Gamma_{\mu\nu}$, which is defined as the limit $M \rightarrow \infty$ of the following integral:

$$\Gamma_{\mu\nu}^{(M^2)} = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left(\frac{(2k+p)_\mu(2k+p)_\nu}{(k^2+m^2)((k+p)^2+m^2)} - (m^2 \mapsto m^2 + M^2) \right) + \delta_{\mu\nu} \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{k^2+m^2} - (m^2 \mapsto m^2 + M^2) \right). \tag{150}$$

Its $(z\bar{z})$ -component may be equivalently rewritten as

$$\Gamma_{z\bar{z}}^{(M^2)} = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left(\frac{(2k+p)_z(2k+p)_{\bar{z}}}{(k^2+m^2)((k+p)^2+m^2)} - 2 \frac{1}{k^2+m^2} \right) - (m^2 \rightarrow m^2 + M^2). \tag{151}$$

Note that the last term, in the limit $M \rightarrow \infty$, is equal to $\int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2+1)^2} = \frac{1}{4\pi}$. Introducing momentarily a factor $e^{ik\epsilon}$ in the (convergent) integral in (151), recalling the definition of $\tau_{z\bar{z}}$ and taking the limit $M \rightarrow \infty$, we may rewrite (151) in the following way:

$$\Gamma_{z\bar{z}} + \frac{1}{4\pi} = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} (\tau_{z\bar{z}}(p|\epsilon) - 2D(\epsilon)) \tag{152}$$

In other words, as ϵ goes to 0, we have the asymptotic behavior

$$\tau_{z\bar{z}}(p|\epsilon) = 2D(\epsilon) - 2 \left(\Gamma_{z\bar{z}} + \frac{1}{4\pi} \right) + \dots \tag{153}$$

Analogously one can show that

$$\tau_{zz}(p|\epsilon) = -2\Gamma_{zz} - \frac{1}{2\pi} \frac{\bar{\epsilon}}{\epsilon} + \dots \tag{154}$$

(For completeness we prove this relation in the Appendix E.)

Substituting the above relations in (149), we arrive at the expression

$$\begin{aligned} & I_{a'}^{(1)} + I_{a'}^{(2)} + I_{b'} + I_{c'} = \\ & = \left(D(\epsilon) - \frac{1}{4\pi} \right) ([\mathcal{A}_z(p_1), \mathcal{A}_{\bar{z}}(p_2)] + [\mathcal{A}_z(p_2), \mathcal{A}_{\bar{z}}(p_1)]) - \\ & - \frac{1}{2} (\Gamma_{z\bar{z}}(p_2) + \Gamma_{z\bar{z}}(p_1)) ([\mathcal{A}_z(p_1), \mathcal{A}_{\bar{z}}(p_2)] + [\mathcal{A}_z(p_2), \mathcal{A}_{\bar{z}}(p_1)]) + \\ & + \frac{1}{2} (\Gamma_{\bar{z}\bar{z}}(p_1) - \Gamma_{\bar{z}\bar{z}}(p_2)) [\mathcal{A}_z(p_1), \mathcal{A}_z(p_2)] + \frac{1}{2} (\Gamma_{zz}(p_1) - \Gamma_{zz}(p_2)) [\mathcal{A}_{\bar{z}}(p_2), \mathcal{A}_{\bar{z}}(p_1)] \end{aligned} \tag{155}$$

We will now argue that, just as we did earlier, the parts proportional to the effective action (i.e. the last two lines) may in fact be dropped. First we rewrite them as

$$-\frac{1}{2} \Gamma_{z\mu}(p_1) [\mathcal{A}_\mu(p_1), \mathcal{A}_{\bar{z}}(p_2)] - \frac{1}{2} \Gamma_{\mu\bar{z}}(p_1) [\mathcal{A}_z(p_2), \mathcal{A}_\mu(p_1)] + (p_1 \leftrightarrow p_2) \tag{156}$$

If one takes for $\mathcal{A}_\mu(p_1)$ the diagonal (gauge) part, say $a_\mu(p_1)$, and computes a correlation function with the field strength operator $(p_1)_\mu a_\nu(p_1) - (p_1)_\nu a_\mu(p_1)$, the effective action part will fully cancel, leaving contributions which in coordinate space are again local (w.r.t. the point where the field strength operator is inserted). Such contributions can be dropped according to the logic explained earlier. On the other hand, if $\mathcal{A}_\mu(p_1)$ is one of the off-diagonal ‘matter’ fields,

say $c_{\bar{z}}(p_1)$, and one computes a correlation function with the insertion of an operator c_z , there will be an additional non-local piece proportional (in momentum space) to $\frac{\Gamma_{z\bar{z}}(p)}{\Gamma_{z\bar{z}}(p)} = -\frac{p_z}{p_{\bar{z}}}$. In the next subsection we will see, however, that in this case there is an additional contribution from the ‘residual part’ in (144) which completely cancels out the $\Gamma_{z\bar{z}}$ -piece in the formula (156).

Anyway (155) is not yet the final answer, as we still have to analyze the diagrams (d') and (e'). The integrand of the diagram (d') has in the numerator the product $k_{\bar{z}}(k_z + p_z)$, for which we can use the decomposition (144). The first two terms can again be used to cancel the propagators, which results in the following contribution (the residual part is again left for a later analysis):

$$\begin{aligned} I_{d'}^{(1)} + I_{d'}^{(2)} &= -\frac{1}{2}D(\epsilon) A_{\mu}(p_1)A_{\mu}(p_2) + (p_1 \leftrightarrow p_2) \\ &= -\frac{1}{2}D(\epsilon) (A_{\bar{z}}(p_1)A_z(p_2) + A_z(p_1)A_{\bar{z}}(p_2)) + (p_1 \leftrightarrow p_2) \end{aligned} \tag{157}$$

The diagram (e') gives $I_{e'} = D(\epsilon)A_{\bar{z}}(p_1)A_z(p_2) + (p_1 \leftrightarrow p_2)$, so that the sum of two diagrams is

$$\begin{aligned} I_{d'}^{(1)} + I_{d'}^{(2)} + I_{e'} &= \frac{1}{2}D(\epsilon) (A_{\bar{z}}(p_1)A_z(p_2) - A_z(p_1)A_{\bar{z}}(p_2)) + (p_1 \leftrightarrow p_2) = \\ &= -\frac{1}{2}D(\epsilon) ([A_z(p_1), A_{\bar{z}}(p_2)] + [A_z(p_2), A_{\bar{z}}(p_1)]) \end{aligned} \tag{158}$$

Together, (155) and (158) amount to (after dropping the Γ -terms)

$$\begin{aligned} I_{a'}^{(1)} + I_{a'}^{(2)} + I_{b'} + I_{c'} + I_{d'}^{(1)} + I_{d'}^{(2)} + I_{e'} &\sim \\ &\sim \left(\frac{1}{2}D(\epsilon) - \frac{1}{4\pi} \right) ([\mathcal{A}_z(p_1), \mathcal{A}_{\bar{z}}(p_2)] + [\mathcal{A}_z(p_2), \mathcal{A}_{\bar{z}}(p_1)]) \end{aligned}$$

Comparing with the leading order OPE (142), we see that the quadratic terms in \mathcal{A} simply amount to completing the linearized field strength to the full non-abelian field strength $F_{z\bar{z}} = \partial_z \mathcal{A}_{\bar{z}} - \partial_{\bar{z}} \mathcal{A}_z + i [\mathcal{A}_z, \mathcal{A}_{\bar{z}}]$.

6.2.4. The residual part

Finally we need to analyze the contributions that the residual part of the decomposition (144) makes to the values of the diagrams (a') and (d'). Our claim is that these contributions cancel partially against the diagrams obtained by the procedure described in Fig. 14, and partially against the Γ -terms in (155), producing contributions that are already included in the linear terms (142) of the OPE. To start with, it is clear that the Feynman integrals corresponding to the diagrams in Fig. 14 are essentially products

$$\text{Lower diagram} \times \text{Connecting propagator} \times \text{Upper diagram} .$$

Since the lower diagrams are the diagrams of Fig. 12, we already know the corresponding expressions: these are given by (139) and (141). If we dropped in the latter expressions the parts with the effective action Γ , we would obtain exactly the correlation functions of the r.h.s. of (142) with two additional \mathcal{A} -operators. Therefore, up to the Γ -terms, the values of the diagrams in Fig. 14 are already included in the linear terms of the OPE (142). As for the Γ -parts of the diagrams on Fig. 14, they cancel with the ‘residual parts’ of the diagrams (a') and (d'), as well as with certain Γ -terms in (155). The mechanism is as follows: the functions $\Gamma_{\lambda}(p)$ and $\Gamma_{z\bar{z}}(p)$ cancel against the corresponding intermediate propagators of Fig. 14, leaving truncated diagrams whose

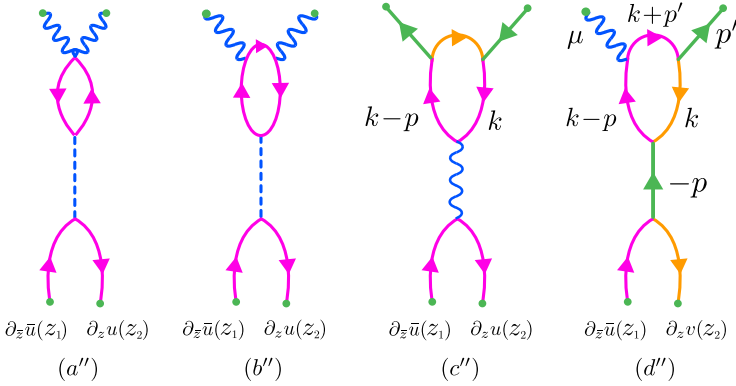


Fig. 14. Some of the additional diagrams at order $\frac{1}{N}$ in the OPE (137), which are obtained by attaching a one-loop (upper) diagram with three external legs to one of the external lines in the diagrams of Fig. 12. The flow of the momentum p is chosen to comply with the convention in Fig. 12 and in the formula (138). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

associated Feynman integrals cancel against the ‘residual parts’ of the diagrams (a’) and (d’), possibly up to contact terms.

Let us elaborate this phenomenon at the example of the rightmost two diagrams in Fig. 14. We will be using the Feynman rules derived in Section 5.

Diagram (c’). The Γ -dependent part of the short-distance expansion of the lower part of the diagram is given by (139) – it is equal to $(p_{\bar{z}}a_z - p_z a_{\bar{z}})\Gamma_{z\bar{z}}$, where a is the $U(1)$ gauge field corresponding to the blue line. Multiplying by the propagators of the gauge fields, we see that the coupling to the upper part of the diagram is given by $(p_{\bar{z}}D_{z\bar{z}}(2k - p)_z + p_{\bar{z}}D_{z\bar{z}}(2k - p)_{\bar{z}} - p_z D_{z\bar{z}}(2k - p)_{\bar{z}} - p_z D_{z\bar{z}}(2k - p)_z)\Gamma_{z\bar{z}}$. Recalling that $D_{z\bar{z}} = \frac{1}{8}\Gamma_{z\bar{z}}^{-1}$ and $D_{z\bar{z}} = -\frac{1}{8}\frac{p_z}{p_{\bar{z}}}\Gamma_{z\bar{z}}^{-1}$, we get $\frac{1}{2}(p_{\bar{z}}k_z - p_z k_{\bar{z}})$. This cancels exactly against the analogous contribution in (144).

Diagram (d’). Just like above, we read off the Γ -dependent part of the short-distance expansion of the lower part of the diagram from (139) – in this case it is equal to $-p_z c_{\bar{z}}\Gamma_{z\bar{z}}$, where $c_{\bar{z}}$ is the off-diagonal auxiliary field in (63). We now supply the propagator of the c -field, which is $D_c = \frac{1}{2}\Gamma_{z\bar{z}}^{-1}$ and the coupling to the upper vertex, which is $(2k - p)_{\bar{z}}$. Altogether this gives $-\frac{1}{2}p_z(2k - p)_{\bar{z}}$. This does not completely cancel the contribution of (144), the sum of the two terms being instead $\frac{1}{4}(p^2 - 2kp) = \frac{1}{4}((k - p)^2 + m^2) - \frac{1}{4}(k^2 + m^2)$. This allows cancelling the propagators in the upper diagram, leading to the following expression:

$$\frac{1}{4} \int \frac{d^2k}{(2\pi)^2} \frac{(2k - p + p')_{\mu} (2k + p')_z}{(k + p')^2 + m^2} \left(\frac{1}{k^2 + m^2} - \frac{1}{(k - p)^2 + m^2} \right) = \frac{1}{2} (\Gamma_{\mu z}(p + p') - \Gamma_{\mu z}(p'))$$

To obtain the latter expression, one needs to subtract the analogous integral with the replacement $m^2 \rightarrow m^2 + M^2$ (which in itself vanishes in the limit $M \rightarrow \infty$), split the resulting integral into two parts, use the definition (150) of $\Gamma_{\mu\nu}^{(M^2)}$ and then take the limit $M \rightarrow \infty$. Now, the above expression is to be multiplied by $a_{\mu}(p + p')c_{\bar{z}}(p')$ and it has to be added to the terms (156) obtained from the diagram (a’) of Fig. 13. In (156) we will assume that $\mathcal{A}_{\mu}(p_1)$ corresponds to

the field $c_{\bar{z}}(p_1)$ (hence $p_1 = p'$) and $A_{\mu}(p_2)$ corresponds to $a_{\mu}(p + p')$ (hence $p_2 = p + p'$). Adding the above expression to (156) and restricting the fields as described, we get

$$\begin{aligned} & \frac{1}{2} (\Gamma_{\mu z}(p_2) - \Gamma_{\mu z}(p_1)) a_{\mu}(p_2) c_{\bar{z}}(p_1) + \\ & + \frac{1}{2} \Gamma_{zz}(p_1) c_{\bar{z}}(p_1) a_{\bar{z}}(p_2) - \frac{1}{2} \Gamma_{z\bar{z}}(p_1) a_z(p_2) c_{\bar{z}}(p_1) - \frac{1}{2} \Gamma_{z\mu}(p_2) a_{\mu}(p_2) c_{\bar{z}}(p_1) = \\ & = -\Gamma_{z\bar{z}}(p_1) a_z(p_2) c_{\bar{z}}(p_1) \end{aligned}$$

This is again a contact term: computing a correlation function with an external c_z -operator produces a local contribution, which may be dropped.

6.3. The anomaly

We are now ready to collect the results that concern the OPE’s of the Noether currents taken at two nearby points obtained in the previous section and calculate the anomaly two-form. To this end, we recall the formulas (117) (the definition of the anomaly) and (143) (the OPE):

$$\begin{aligned} \mathcal{Q}_{\epsilon}(\Gamma_1) - \mathcal{Q}_{\epsilon}(\Gamma_2) &= \int_{D_{\delta\Gamma}} dz \wedge d\bar{z} \left[\left(\text{Re}(\mathbf{a}) + \frac{1}{2\pi} \log |\epsilon| \right) (\partial_z K_{\bar{z}} - \partial_{\bar{z}} K_z) - \right. \\ & \quad \left. - \frac{1}{2g^2} ([K_z(p + \epsilon), K_{\bar{z}}(p)] + [K_z(p), K_{\bar{z}}(p + \epsilon)]) \right]. \\ [K_z(z_1), K_{\bar{z}}(z_2)]^{ij} &\xrightarrow{z_2 \rightarrow z_1} g^2 \left([\partial_{\bar{z}} K_z - \partial_z K_{\bar{z}}] D(\epsilon) + \frac{g^2}{\pi N} \varphi i F_{z\bar{z}} \varphi^{\dagger} \right)_{z_1}^{ij} + \dots \end{aligned}$$

Since, according to (119), $D(\epsilon) = -\frac{1}{4\pi} \log \left(\frac{m^2 |\epsilon|^2}{2} \right) - \frac{\gamma}{2\pi} + \dots$, we will set $\text{Re}(\mathbf{a}) = \frac{1}{4\pi} \log \left(\frac{m^2}{2} \right) + \frac{\gamma}{2\pi}$. Then in the limit $\epsilon \rightarrow 0$ we obtain the following result:

$$\mathcal{Q}_{\epsilon}(\Gamma_1) - \mathcal{Q}_{\epsilon}(\Gamma_2) = \int_{D_{\delta\Gamma}} dz \wedge d\bar{z} : - \frac{g^4}{\pi N} \varphi i F_{z\bar{z}} \varphi^{\dagger} : , \tag{159}$$

where $F_{z\bar{z}} = \partial_z \mathcal{A}_{\bar{z}} - \partial_{\bar{z}} \mathcal{A}_z + i [\mathcal{A}_z, \mathcal{A}_{\bar{z}}]$,

and the ‘gauge field’ \mathcal{A} is in general of the restricted form (80) (for the particular case of the flag manifold (61) it is of the form (63)). Note that formally all our calculations were performed for the off-diagonal matrix elements ($i \neq j$), therefore since $\mathcal{Q}_{\epsilon} \in \mathfrak{su}_N$, one would need to subtract the trace-part of the respective expressions (which is proportional to the identity matrix $\mathbb{1}_N$). Instead, we have inserted the normal ordering in the r.h.s. of (159), as in this case the trace-part vanishes due to the property $:\varphi^{\dagger}(z)\varphi(z): := 0$ discussed earlier in Section 6.2.2 (see Fig. 9).

7. Conclusion and outlook

In the present paper we investigated the classical and quantum properties of the flag manifold sigma-models, introduced in our work [31,32]. In fact, the models of [31] constitute a slightly broader class, allowing for complex homogeneous target spaces, however the subclass of flag manifold models is representative and interesting enough for an elaborate study. As we showed in Section 3, models of this class may also be understood in terms of the theory of sigma-models

with \mathbb{Z}_m -symmetric target spaces considered in [24]. Throughout the paper for simplicity we restricted to the case when the isometry group is $SU(N)$, although we expect that similar analysis might be carried out for other compact semi-simple groups. With the aim of constructing the $\frac{1}{N}$ -expansion for the flag manifold sigma-models, we first reformulated these models as gauged linear sigma-models in Section 4. The construction is non-trivial, as the target spaces in question are not Kähler. Nevertheless, formally the result is akin to the GLSM-formulation for Grassmannian (Kähler) target spaces, albeit with a ‘gauge field’ of a special (restricted) form. In the next section, using this formulation, we derived the Feynman rules for the simplest non-symmetric-space model. Subsequently we introduced a regularized version of the non-local charge, which is different from the original definition of Lüscher [15]. As opposed to the definition in [15], our version of the regularized charge depends on the integration contour Γ through an anomaly two-form and does not depend on the parametrization of the contour. We compute this anomaly two-form explicitly in Sections 6.2, 6.3. It formally coincides with the anomaly for the Grassmannian models [18], but the gauge fields entering the anomaly should be taken in restricted form, of the type mentioned earlier. It is an important and interesting question, how the anomaly may be cancelled. Most likely this can be done by introducing fermions, i.e. by a mechanism similar to the one of [19,22]. The investigation of the peculiarities of this mechanism in application to the present models will be a subject of future work.

Acknowledgements

I would like to thank I. Aref’eva, S. Frolov, E. Ivanov, S. Ketov, O. Lechtenfeld, A. Maltsev, M. Semenov-Tian-Shansky, A. Tseytlin, K. Zarembo, P. Zinn-Justin for valuable comments. I am indebted to Prof. A.A. Slavnov and to my parents for support and encouragement. I would like to thank the Institut des Hautes Études Scientifiques, where part of this work was done, and in particular V. Pestun for hospitality.

Appendix A. Flag manifolds and elements of representation theory

This Appendix lies somewhat outside the main line of exposition in the present paper. Yet we have included it to demonstrate, how flag manifolds arise in a well-known physical situation. Incidentally this makes a neat connection to the applications of flag manifolds in representation theory, discussed below in Section A.1.

It is well-known, how one can describe a classical particle, interacting with an external electromagnetic field A_μ . The action has the form

$$S = \int dt \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{2} - \int A = \int dt \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{2} - A_\mu \dot{x}^\mu \right). \tag{160}$$

The question is, how to write an analogous action for the case when the gauge field is non-abelian, or, simply speaking, when it has additional gauge indices A_μ^{ab} . The answer is that the particle should possess additional degrees of freedom, taking values in a certain flag manifold, corresponding to the representation, in which the particle transforms. In other words, one should enlarge the phase space as follows [57] (here \mathcal{M} is the configuration space):

$$T^* \mathcal{M} \rightarrow T^* \mathcal{M} \times \mathcal{F} \text{ (the flag manifold)}. \tag{161}$$

We start by rewriting the standard action of a particle in first-order form:

$$S = \int dt \left(p_\mu \dot{x}^\mu - \frac{g^{\mu\nu} p_\mu p_\nu}{2} - A_\mu \dot{x}^\mu \right) \tag{162}$$

Upon enlarging the phase space we can analogously write down the non-abelian action as follows (A is assumed Hermitian):

$$S = \int p_\mu dx^\mu - \int dt \mathcal{H}(x, p) + \int (\theta - \text{tr}(A \mu)), \tag{163}$$

where θ is the canonical one-form, defined by the condition

$$d\theta = \Omega \quad (= \text{the symplectic form on } \mathcal{F}), \tag{164}$$

and μ is the moment map for the action of the group G on \mathcal{F} . We note that the form θ is only defined up to the addition of a total derivative, $\theta \rightarrow \theta + dh$, but the difference only affects the boundary terms in the action. In the case of periodic boundary conditions one may even write

$$\int_\Gamma \theta = \int_D \Omega, \tag{165}$$

where D is a disc, whose boundary is the curve $\Gamma: \partial D = \Gamma$. In fact this term is nothing but the one-dimensional version of the Wess–Zumino–Novikov–Witten term [58–60].

One needs to show that the expression so obtained is gauge-invariant. For simplicity let us consider the standard representation of $SU(N)$ (when the representation space is \mathbb{C}^N): in this case the relevant flag manifold is the projective space $\mathbb{C}P^{N-1}$. Let us normalize the homogeneous coordinates (z_1, \dots, z_N) on $\mathbb{C}P^{N-1}$:

$$\sum_{k=1}^N |z_k|^2 = 1. \tag{166}$$

One still has the remaining gauge group $U(1)$, which acts by multiplication of all coordinates z_k by a common phase. The Fubini–Study form on $\mathbb{C}P^{N-1}$, when written in homogeneous coordinates, looks as follows:

$$\omega_{FS} = \frac{i}{\|z\|^2} \left(dz_k \wedge d\bar{z}_k - \frac{\bar{z}_n dz_n \wedge z_m d\bar{z}_m}{\|z\|^2} \right), \tag{167}$$

but it may be simplified, if one uses the above normalization:

$$\omega_{FS} = i dz_k \wedge d\bar{z}_k. \tag{168}$$

Then we have the following expressions for θ and μ :

$$\theta = i z_k d\bar{z}_k, \quad \mu = z \otimes \bar{z} - \frac{1}{N} \mathbb{1}_N. \tag{169}$$

The part of the action, corresponding to the motion in the ‘internal’ space (in this case the projective space), has the form

$$\tilde{S} = \int dt \left(i z_k \dot{\bar{z}}_k - \bar{z}_m (A_\mu)_{mn} \dot{x}^\mu z_n \right) = \tag{170}$$

$$= - \int dt \bar{z}_m (i \dot{z}_m + (A_\mu)_{mn} \dot{x}^\mu z_n), \tag{171}$$

and one should take into account that the normalization condition (166) is also implied. From the second form of the action it is evident that it is gauge-invariant w.r.t. the transformations

$$z \rightarrow g(x(t)) \circ z \quad A_\mu \rightarrow g A_\mu g^{-1} - i \partial_\mu g g^{-1}. \tag{172}$$

To make it even more obvious, we note that the exterior derivative of the one-form $\theta - \text{tr}(A \mu)$ (viewed as a form on the enlarged phase space (161)) produces a two-form, which is explicitly gauge-invariant:

$$\begin{aligned} d(\theta - \text{tr}(A \mu)) &= i \mathcal{D}z_k \wedge \mathcal{D}\bar{z}_k - \text{tr}(F \mu), \\ \mathcal{D}z &= dz - i A z, \quad \mathcal{D}\bar{z} = d\bar{z} + i \bar{z} A, \quad F = dA - i A \wedge A. \end{aligned} \tag{173}$$

Each of the two terms in (173) is separately gauge-invariant, however (173) is the only linear combination of them, which is closed (and therefore locally is an exterior derivative of a one-form).

Next we wish to write out the equations of motion on the flag manifold, which follow from the action above. Let us concentrate for simplicity on the $SU(2)$ -case, where the sphere S^2 plays the role of a flag manifold. Instead of using the spinor (z_1, z_2) , we can parametrize it in a more standard way, with the help of a unit vector $\vec{n} \in \mathbb{R}^3$. The equations take the form

$$\dot{\vec{n}} = \vec{A} \times \vec{n}, \quad \text{where} \quad \vec{A} = \{A_\mu^a \dot{x}^\mu\}_{a=1,2,3} \tag{174}$$

is a vector of components of the gauge field in the basis of Pauli matrices. We see that the equations are *linear* in \vec{n} , and the condition

$$\vec{n}^2 = \text{const.} \tag{175}$$

itself is a consequence of the equations, i.e. the motion takes place on a sphere in \mathbb{R}^3 . This is a general fact. Indeed, in the case of a general compact simple Lie algebra \mathfrak{g} with basis $\{\tau_i\}$ we can introduce a variable $n = \sum x^i \tau_i \in \mathfrak{g}$, and the equations will then take the form

$$\dot{n} = [A_\mu \dot{x}^\mu, n], \tag{176}$$

or, in terms of the variables x^i ,

$$\dot{x}^i = f_{jk}^i (A_\mu \dot{x}^\mu)^j x^k. \tag{177}$$

It is in this form that this system of equations was discovered in [61]. The motion defined by these equations in reality takes place on flag manifolds embedded in \mathfrak{g} , since the ‘Casimirs’

$$C_J = \text{tr}(n^J), \quad J = 1, 2, \dots \tag{178}$$

are integrals of motion of the system (176). We have thus established a connection with the formulation through flag manifolds used earlier.

A.1. ‘Quantization’ of the symplectic form on flag manifolds

One of the approaches to quantization is related to considering path integrals of the following form⁷:

⁷ Another approach to the quantization of coadjoint orbits, which is also based on the path integral, was developed in [62].

$$\int \prod_i d\varphi_i e^{i\mathcal{S}} \tag{179}$$

where the exponent contains the action (163). The connection θ is not a globally-defined one-form on the flag manifold. Indeed, let us consider the simplest case of $\mathcal{F} = \mathbb{C}\mathbb{P}^1 = S^2$. The most general invariant symplectic form is as follows:

$$\omega = \frac{a}{2} \sin \vartheta d\vartheta \wedge d\phi \tag{180}$$

with an arbitrary constant a . It can be also written in the form $\omega = -\frac{a}{2} dz \wedge d\phi$, where $z = \cos \vartheta$ is the z -coordinate of a given point on the sphere. Since the latter form is nothing but the area element of a cylinder, it implies that the projection of a sphere to the cylinder preserves the area. Since the action \mathcal{S} entering the exponent in (179) involves a term $\int \theta$, where θ is a connection satisfying $d\theta = \omega$, standard arguments familiar from Wess–Zumino–Novikov–Witten theory [58–60] lead to the requirement that the coefficient a is quantized according to $\int \omega \in 2\pi n, n \in \mathbb{Z}$. By analogy, in the case when the manifold \mathcal{F} has several non-trivial basic cycles

C_1, \dots, C_{N-1} , in order for the exponent $e^{i \int_{\Gamma} \theta}$ to be well-defined for any contour Γ , the integral of the symplectic form Ω over any 2-cycle C_i (which in the case of a flag manifold is always a sphere S^2) should be quantized:

$$\int_{C_i} \Omega \in 2\pi \mathbb{Z} \quad \text{for every 2-cycle } C_i \in H_2(\mathcal{F}, \mathbb{Z}). \tag{181}$$

Let us construct these 2-cycles explicitly for the case when \mathcal{F} is a complete flag manifold

$$\mathcal{F}(1, \dots, N-1, N) = \{0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^N\} = \frac{U(N)}{U(1)^N}. \tag{182}$$

It can be parametrized using N orthonormal vectors $u_i, i = 1 \dots N, u_i \circ \bar{u}_j = \delta_{ij}$, defined modulo phase transformations: $u_k \sim e^{i\alpha_k} u_k$. As we showed in Section 1.2, the most general symplectic form on $\mathcal{F}(1, \dots, N-1, N)$ may be written as follows:

$$\Omega = i \sum_{i < j} a_{ij} J_{ij} \wedge J_{ji}, \quad \text{where } J_{ij} = u_i \circ d\bar{u}_j \tag{183}$$

$$\text{and } a_{ij} = z_i - z_j. \tag{184}$$

If one fixes $N - 2$ out of N lines defined by the vectors u_1, \dots, u_N , the remaining free parameters define the configuration space of ordered pairs of mutually orthogonal lines, passing through the origin and laying in a plane, orthogonal to the $N - 2$ fixed lines. This configuration space is nothing but the sphere $\mathbb{C}\mathbb{P}^1$:

$$\{u_{i_1}, \dots, u_{i_{N-2}} \text{ are fixed, } u_{i_{N-1}}, u_{i_N} \in (u_{i_1}, \dots, u_{i_{N-2}})^\perp \\ \text{are mutually orthogonal and otherwise generic}\} \simeq (\mathbb{C}\mathbb{P}^1)_{i_{N-1}, i_N}.$$

Let us now fix the permutation (i_1, \dots, i_N) in such a way that z_{i_m} would form a non-increasing sequence, i.e. $z_{i_m} \geq z_{i_n}$ for $m < n$ (in the case of the complete flag manifold (182) the sequence should be strictly decreasing, whereas the ‘non-increasing’ case in general corresponds to partial flag manifolds in a natural way). We can rewrite the symplectic form as follows: $\Omega = i \sum_{m < n} a_{i_m i_n} J_{i_m i_n} \wedge J_{i_n i_m}$, and uniquely fix a complex structure, in which the one-forms $J_{i_m i_n}, m < n$, are holomorphic. After such a permutation we may choose $(\mathbb{C}\mathbb{P}^1)_{i_m, i_{m+1}}$ as

a basis in the homology group $H_2(\mathcal{F}_{1,\dots,N-1,N}, \mathbb{Z})$, with the orientation of the spheres induced by the complex structure. Then the integrals of the symplectic form over these cycles will be positive:

$$\int_{(\mathbb{C}\mathbb{P}^1)_{i_m, i_{m+1}}} \Omega = z_{i_m} - z_{i_{m+1}} \in 2\pi\mathbb{Z}^+, \quad m = 1 \dots N - 1. \tag{185}$$

In order for the value of the integral to be an integer, one should choose z_i in the form

$$(z_1, \dots, z_N) = \lambda(1, \dots, 1) + (\tilde{z}_1, \dots, \tilde{z}_N), \quad \lambda \in \mathbb{R}, \quad \tilde{z}_i \in \mathbb{Z}. \tag{186}$$

We may view $(\mathbb{C}\mathbb{P}^1)_{i_m, i_{m+1}}$ as the positive simple roots, and Ω is, in this case, a dominant weight. Adding to $\{\tilde{z}_i\}$ a vector, proportional to $\lambda(1, \dots, 1)$, does not change the values of the integrals. Using this property, we may normalize the values $\{\tilde{z}_i\}$ in such a way that their sum would be zero. According to the general theory of adjoint orbits, the flag manifold under consideration is the orbit of the element

$$z = \begin{pmatrix} \tilde{z}_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \tilde{z}_N \end{pmatrix} \in \mathfrak{su}_N. \tag{187}$$

The fundamental weights are, by definition, the ones, whose value on simple positive roots is equal to zero, apart from a single root, on which the fundamental weight has value one: $\langle \Omega_{F_{[i]}}, \lambda_{[i]} \rangle = \delta_{ij}$. These weights correspond to the highest weights of fundamental representations. According to the theory described above, they correspond to orbits of the elements

$$z_{F_{[i]}} = \text{Diag}(\underbrace{1, \dots, 1}_i, \underbrace{0, \dots, 0}_{N-i}). \tag{188}$$

According to (185), in this case we have

$$\int_{(\mathbb{C}\mathbb{P}^1)_{k, k+1}} \Omega_{F_{[i]}} = \delta_{ik}. \tag{189}$$

The adjoint orbit in question is the Grassmannian $G_{i,N}$. The general theory that we have described is nothing but ‘geometric quantization’ for the case of flag manifolds.

Appendix B. Eliminating auxiliary fields

In this Appendix we demonstrate explicitly how the auxiliary fields a, b, c may be eliminated from the Lagrangian (60) with $m + n = 2$ and the gauge field given by (63). To this end, we write the Lagrangian as follows:

$$\begin{aligned} \mathcal{L} = & \partial_{\bar{z}}\bar{u} \circ \partial_z u + \partial_{\bar{z}}\bar{v} \circ \partial_z v + \\ & + \frac{N}{g^2} a_{\bar{z}} a_z + a_{\bar{z}}(i \bar{u} \circ \partial_z u) - a_z(i \partial_{\bar{z}}\bar{u} \circ u) + \\ & + \frac{N}{g^2} b_{\bar{z}} b_z + b_{\bar{z}}(i \bar{v} \circ \partial_z v) - b_z(i \partial_{\bar{z}}\bar{v} \circ v) + \\ & + \frac{N}{g^2} c_{\bar{z}} c_z + c_{\bar{z}}(i \bar{v} \circ \partial_z u) - c_z(i \partial_{\bar{z}}\bar{u} \circ v) \end{aligned} \tag{190}$$

Here we have taken advantage of the orthonormality relations $\|u\| = \|v\| = \frac{\sqrt{N}}{g}$ and $\bar{u} \circ v = 0$. Completing the squares, we can obtain relations of the form

$$\frac{N}{g^2} a_{\bar{z}} a_z + a_{\bar{z}}(i \bar{u} \circ \partial_z u) - a_z(i \partial_{\bar{z}} \bar{u} \circ u) = -\frac{g^2}{N} (\bar{u} \circ \partial_z u)(\partial_{\bar{z}} \bar{u} \circ u) + \text{complete square}. \quad (191)$$

Therefore upon setting the fields a, b, c equal to their stationary values, we get the Lagrangian in the form (64).

Appendix C. The point splitting of Lüscher

In this section we recall the point splitting method for the non-local charge, introduced in the original paper [15]. In order to measure this splitting one needs to introduce a metric g on the worldsheet Σ_2 (note that classically everything depended only on the conformal class $[g]$ of the metric). For simplicity we will now consider the situation when $\Sigma_2 \simeq \mathbb{R}^2$, equipped with a flat metric. As in the body of the paper, we will assume that the contour Γ is infinite (a straight line, for example) and we will fix the reparametrization invariance on the worldline by choosing a gauge

$$(\dot{x}^\mu)^2 = 1. \quad (192)$$

Here $x^\mu : \Gamma \simeq \mathbb{R}_t \rightarrow \Sigma_2$ is the parametrization of Γ by a variable t taking values from minus infinity to plus infinity. If Γ was a contour of finite length, one could still parametrize it by a variable t with infinite range, however one would need to choose a gauge $(\dot{x}^\mu)^2 = f^2(t)$, so that the length $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ was finite.

The regularized non-local charge of [15] has the form:

$$Q_2^\epsilon = Z(\epsilon) \int_{-\infty}^{\infty} dt \dot{x}^\mu K_\mu - \frac{1}{2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{s-\epsilon} dt [(*K)_t, (*K)_s], \quad (193)$$

where ϵ is now a positive real number. The factor $Z(\epsilon)$ has been inserted to cancel the divergence in the short-distance expansion of the two currents in the second term. Next we compute the variation of this charge under deformations of the contour δx^μ (sometimes we will denote this variation by δ_Γ). Upon integrating by parts in the terms involving $\delta \dot{x}^\mu$, we obtain:

$$\begin{aligned} \delta_\Gamma Q_2^\epsilon &= Z(\epsilon) \int_{-\infty}^{\infty} dt \delta x^\mu \dot{x}^\nu (\partial_\mu K_\nu - \partial_\nu K_\mu) - \\ &- \frac{1}{2} \int_{-\infty}^{\infty} dt \delta x^\mu(t - \epsilon) \dot{x}^\alpha(t) [(*K)_\mu(t - \epsilon), (*K)_\alpha(t)] + \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} dt \dot{x}^\mu(t) \delta x^\alpha(t + \epsilon) [(*K)_\mu(t), (*K)_\alpha(t + \epsilon)] + \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{s-\epsilon} dt ((\delta x^\mu(t) \dot{x}^\lambda(t) - \delta x^\lambda(t) \dot{x}^\mu(t)) [\partial_\lambda (*K)_\mu(t), (*K)_\alpha(s)] \dot{x}^\alpha(s) - (s \leftrightarrow t)) \end{aligned}$$

The ‘bulk term’ (the one in the last line) vanishes, since due to the antisymmetry of $\delta x^\mu(t) \dot{x}^\lambda(t) - \delta x^\lambda(t) \dot{x}^\mu(t)$ it turns out to be proportional to $\partial_\mu K_\mu = 0$ (conservation of the current). The final result is therefore

$$\delta_\Gamma Q_2^\epsilon = \int_{-\infty}^{\infty} dt [Z(\epsilon) \delta x^\mu \dot{x}^\nu (\partial_\mu K_\nu - \partial_\nu K_\mu) + \frac{1}{2} \epsilon_{\mu\nu} \epsilon_{\alpha\beta} [K_\nu(t - \epsilon), K_\beta(t)] (\dot{x}^\mu(t - \epsilon) \delta x^\alpha(t) - \delta x^\mu(t - \epsilon) \dot{x}^\alpha(t))] \tag{194}$$

Now in the last line we will use the OPE's of two currents obtained in Section 6.2. If, say, $\mu = z$ and $\alpha = \bar{z}$, one needs the OPE $[K_z(p + \epsilon), K_{\bar{z}}(p)]$, which, according to (143), is logarithmic in ϵ . Therefore in this case in the bracket $(\dot{z}(t - \epsilon) \delta \bar{z}(t) - \delta z(t - \epsilon) \dot{\bar{z}}(t))$ we can pass to the limit $\epsilon \rightarrow 0$. The logarithmic divergence itself is cancelled by tuning the dependence on ϵ of the factor $Z(\epsilon)$.

If however $\mu = \alpha = z$, we have to consider the OPE $[K_{\bar{z}}(p + \epsilon), K_{\bar{z}}(p)]$, which according to (124) has a pole term $\sim \frac{1}{\epsilon}$ at the leading order. This term is multiplied by $(\dot{z}(t - \epsilon) \delta z(t) - \delta z(t - \epsilon) \dot{z}(t)) \sim \epsilon (\delta z \dot{z} - \dot{z} \delta z)$, producing an undesirable extra contribution in addition to (116). In other words, if one assumes the definition (193), the anomaly is not given by a 2-form.

Appendix D. The integral giving the anomaly at order $\frac{1}{\sqrt{N}}$

In this Appendix we wish to consider the integral (138):

$$I = \int \frac{d^2k}{(2\pi)^2} \left(\frac{k_{\bar{z}}(k_z + p_z)}{(k^2 + m^2)((k + p)^2 + m^2)} ((2k_{\bar{z}} + p_{\bar{z}}) \mathcal{A}_z(p) + (2k_z + p_z) \mathcal{A}_{\bar{z}}(p)) - \frac{k_{\bar{z}}}{k^2 + m^2} \mathcal{A}_z(p) - \frac{k_z + p_z}{(p + k)^2 + m^2} \mathcal{A}_{\bar{z}}(p) \right) e^{ik\epsilon}$$

and to bring it to the form (139). We start by rewriting the factors in the numerators of the first two summands as follows:

$$k_{\bar{z}}(k_z + p_z) = \frac{1}{2}((k + p)^2 + m^2) - p_{\bar{z}}(k_z + p_z) - \frac{1}{2}m^2 \tag{195}$$

$$k_{\bar{z}}(k_z + p_z) = \frac{1}{2}(k^2 + m^2) + k_{\bar{z}}p_z - \frac{1}{2}m^2 \tag{196}$$

The first terms here allow cancelling one of the propagators each. The remaining terms are multiplied by a factor $e^{ik\epsilon} \left(\frac{2k_\mu + p_\mu}{(k^2 + m^2)((k + p)^2 + m^2)} \right)$, which in the limit $\epsilon \rightarrow 0$ is odd under the interchange $k \rightarrow -k - p$. As a result, we are free to add to (195)–(196) arbitrary terms that are independent of the momentum k , since such terms will vanish in the integral in the limit $\epsilon \rightarrow 0$ due to antisymmetry. We use this arbitrariness to bring the expressions (195)–(196) to the form

$$k_{\bar{z}}(k_z + p_z) \rightarrow \frac{1}{2}((k + p)^2 + m^2) - \frac{1}{2}p_{\bar{z}}(2k_z + p_z) \tag{197}$$

$$k_{\bar{z}}(k_z + p_z) \rightarrow \frac{1}{2}(k^2 + m^2) + \frac{1}{2}(2k_{\bar{z}} + p_{\bar{z}})p_z. \tag{198}$$

When substituted into the integral, this gives

$$I = \int \frac{d^2k}{(2\pi)^2} \left[p_z \mathcal{A}_{\bar{z}} \left(\frac{1}{2} \frac{(2k_z + p_z)(2k_{\bar{z}} + p_{\bar{z}})}{(k^2 + m^2)((k + p)^2 + m^2)} - \frac{1}{2} \frac{1}{(k + p)^2 + m^2} \right) - p_{\bar{z}} \mathcal{A}_z \left(\frac{1}{2} \frac{(2k_z + p_z)(2k_{\bar{z}} + p_{\bar{z}})}{(k^2 + m^2)((k + p)^2 + m^2)} - \frac{1}{2} \frac{1}{k^2 + m^2} \right) \right] \times e^{ik\epsilon} + \dots$$

Again, up to terms that vanish in the limit $\epsilon \rightarrow 0$ (which arise due to the change of variables $k \rightarrow -k - p$ in the first bracket), we obtain:

$$I = (p_z A_{\bar{z}} - p_{\bar{z}} A_z) \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{2} \frac{(2k_z + p_z)(2k_{\bar{z}} + p_{\bar{z}})}{(k^2 + m^2)((k + p)^2 + m^2)} - \frac{1}{2} \frac{1}{k^2 + m^2} \right) \times e^{ik\epsilon} + \dots$$

We now wish to relate this integral to the $\Gamma_{z\bar{z}}$ function entering the effective action (90). The analogous function $\Gamma_{z\bar{z}}^{(M^2)}$ in the presence of a Pauli–Villars regulator of mass M can be written as a difference of two convergent terms:

$$\Gamma_{z\bar{z}}^{(M^2)} = \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{k^2 + m^2} - \frac{1}{2} \frac{(2k_z + p_z)(2k_{\bar{z}} + p_{\bar{z}})}{(k^2 + m^2)((k + p)^2 + m^2)} \right) - (m^2 \rightarrow m^2 + M^2) \tag{199}$$

The limit of the last term ($m^2 \rightarrow m^2 + M^2$) as $M \rightarrow \infty$ is $\frac{1}{4\pi}$. Therefore we can rewrite I as follows:

$$I = (p_{\bar{z}} A_z - p_z A_{\bar{z}}) \left(\Gamma_{z\bar{z}} + \frac{1}{4\pi} \right) - \frac{1}{2} (p_{\bar{z}} A_z - p_z A_{\bar{z}}) D(\epsilon) + \dots, \tag{200}$$

which is the expression (139).

Appendix E. One more integral

Here we will prove the relation

$$\tau_{zz}(p|\epsilon) = -2\Gamma_{zz} - \frac{1}{2\pi} \frac{\bar{\epsilon}}{\epsilon} + \dots \quad \text{for } \epsilon \rightarrow 0 \tag{201}$$

between the Γ_{zz} -component of the effective action and the integral

$$\tau_{zz}(p|\epsilon) := \int \frac{d^2k}{(2\pi)^2} e^{i(k \cdot \epsilon)} \frac{(2k + p)_z (2k + p)_{\bar{z}}}{((k + p)^2 + m^2)(k^2 + m^2)}, \tag{202}$$

which we encountered in Section 6.2.3. First, we formally introduce a factor $e^{ik\epsilon}$ in the definition of Γ_{zz} , which allows us to write:

$$\Gamma_{zz} = \lim_{\epsilon \rightarrow 0} \left(-\frac{1}{2} \tau_{zz}(p|\epsilon) + \underbrace{\int \frac{d^2k}{(2\pi)^2} \frac{2k_z^2 e^{i(k \cdot \epsilon)}}{(k^2 + m^2)^2}}_{:=\mathcal{I}} \right) \tag{203}$$

Therefore we need to construct the asymptotics of the second (simpler) integral \mathcal{I} as $\epsilon \rightarrow 0$. We recall that, by definition, $(k \cdot \epsilon) = k_z \epsilon + k_{\bar{z}} \bar{\epsilon}$. As a consequence of rotational symmetry $\epsilon \rightarrow e^{i\gamma} \epsilon$, the integral is equal to

$$\mathcal{I} = \frac{\bar{\epsilon}}{\epsilon} f(|\epsilon|). \tag{204}$$

To calculate f , we may set ϵ to be real and positive. We pass to polar coordinates setting $k_z = \frac{\rho}{\sqrt{2}} e^{i\psi}$, then $f = \int_0^\infty \frac{\rho d\rho}{2\pi} \frac{\rho^2}{(\rho^2 + m^2)^2} \int_0^{2\pi} \frac{d\psi}{2\pi} e^{i(2\psi + \rho \epsilon \cos(\psi))} = - \int_0^\infty \frac{\rho d\rho}{2\pi} \frac{\rho^2 J_2(\rho\epsilon)}{(\rho^2 + m^2)^2}$. Rescaling $\rho \rightarrow \frac{\rho}{\epsilon}$ and passing to the limit $\epsilon \rightarrow 0$ (which does not introduce divergences in the integral, thanks to

the properties of the Bessel function at 0 and ∞), we get $\lim_{\epsilon \rightarrow 0} f(\epsilon) = -\int_0^{\infty} \frac{d\rho}{2\pi\rho} J_2(\rho) = -\frac{1}{4\pi}$.

This leads to the relation (201).

References

- [1] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Monographs in Contemporary Mathematics, Springer US, 1984.
- [2] K. Pohlmeyer, Integrable Hamiltonian systems and interactions through quadratic constraints, Commun. Math. Phys. 46 (1976) 207.
- [3] V.E. Zakharov, A.V. Mikhailov, Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique, Sov. Phys. JETP 47 (1978) 1017, Zh. Eksp. Teor. Fiz. 74 (1978) 1953.
- [4] K. Uhlenbeck, Harmonic maps into Lie groups (classical solutions of the chiral model), J. Differ. Geom. 30 (1989) 1–50.
- [5] N.J. Hitchin, Harmonic maps from a 2-torus to the 3-sphere, J. Differ. Geom. 31 (1990) 627–710.
- [6] I. Arefeva, V. Korepin, Scattering in two-dimensional model with Lagrangian $\frac{1}{\gamma}((\partial_\mu u)^2/2 + m^2(\cos u - 1))$, Pis'ma Zh. Eksp. Teor. Fiz. 20 (1974) 680.
- [7] A.B. Zamolodchikov, Exact two-particle S matrix of quantum solitons of the sine-Gordon model, JETP Lett. 25 (1977) 468.
- [8] A.B. Zamolodchikov, A.B. Zamolodchikov, Factorized s matrices in two-dimensions as the exact solutions of certain relativistic quantum field models, Ann. Phys. 120 (1979) 253.
- [9] K. Zarembo, Integrability in sigma-models, arXiv:1712.07725 [hep-th].
- [10] A. D'Adda, M. Lüscher, P. Di Vecchia, A $1/n$ expandable series of nonlinear sigma models with instantons, Nucl. Phys. B 146 (1978) 63.
- [11] Y.Y. Goldschmidt, E. Witten, Conservation laws in some two-dimensional models, Phys. Lett. B 91 (1980) 392.
- [12] A.M. Polyakov, Hidden symmetry of the two-dimensional chiral fields, Phys. Lett. B 72 (1977) 224.
- [13] J.M. Maillet, Kac–Moody algebra and extended Yang–Baxter relations in the $O(N)$ nonlinear σ model, Phys. Lett. B 162 (1985) 137.
- [14] J.M. Maillet, New integrable canonical structures in two-dimensional models, Nucl. Phys. B 269 (1986) 54.
- [15] M. Lüscher, Quantum nonlocal charges and absence of particle production in the two-dimensional nonlinear sigma model, Nucl. Phys. B 135 (1978) 1.
- [16] D. Bernard, Hidden Yangians in 2-D massive current algebras, Commun. Math. Phys. 137 (1991) 191.
- [17] E. Abdalla, M.C.B. Abdalla, M. Gomes, Anomaly in the nonlocal quantum charge of the $CP^{(n-1)}$ model, Phys. Rev. D 23 (1981) 1800.
- [18] E. Abdalla, M. Forger, M. Gomes, On the origin of anomalies in the quantum nonlocal charge for the generalized nonlinear σ models, Nucl. Phys. B 210 (1982) 181.
- [19] E. Abdalla, M. Forger, Integrable nonlinear σ models with fermions, Commun. Math. Phys. 104 (1986) 123.
- [20] E. Cremmer, J. Scherk, The supersymmetric nonlinear sigma model in four-dimensions and its coupling to supergravity, Phys. Lett. B 74 (1978) 341.
- [21] A. D'Adda, P. Di Vecchia, M. Lüscher, Confinement and chiral symmetry breaking in CP^{n-1} models with quarks, Nucl. Phys. B 152 (1979) 125.
- [22] E. Abdalla, M.C.B. Abdalla, M. Gomes, Anomaly cancellations in the supersymmetric $CP^{(N-1)}$ model, Phys. Rev. D 25 (1982) 452.
- [23] I. Bena, J. Polchinski, R. Roiban, Hidden symmetries of the $AdS_5 \times S^5$ superstring, Phys. Rev. D 69 (2004) 046002.
- [24] C.A.S. Young, Non-local charges, $Z(m)$ gradings and coset space actions, Phys. Lett. B 632 (2006) 559.
- [25] M. Guest, Harmonic Maps, Loop Groups, and Integrable Systems, London Mathematical Society Student Texts, Cambridge University Press, 1997.
- [26] I.V. Cherednik, Relativistically invariant quasiclassical limits of integrable two-dimensional quantum models, Theor. Math. Phys. 47 (1981) 422, Teor. Mat. Fiz. 47 (1981) 225.
- [27] V.A. Fateev, The sigma model (dual) representation for a two-parameter family of integrable quantum field theories, Nucl. Phys. B 473 (1996) 509.
- [28] C. Klimcik, On integrability of the Yang–Baxter sigma-model, J. Math. Phys. 50 (2009) 043508.
- [29] F. Delduc, S. Lacroix, M. Magro, B. Vicedo, On the Hamiltonian integrability of the bi-Yang–Baxter sigma-model, J. High Energy Phys. 1603 (2016) 104.
- [30] S. Lacroix, Integrable models with twist function and affine Gaudin models, arXiv:1809.06811 [hep-th].

- [31] D. Bykov, Complex structures and zero-curvature equations for σ -models, Phys. Lett. B 760 (2016) 341, arXiv:1605.01093 [hep-th].
- [32] D. Bykov, Integrable properties of sigma-models with non-symmetric target spaces, Nucl. Phys. B 894 (2015) 254.
- [33] D. Bykov, Classical solutions of a flag manifold σ -model, Nucl. Phys. B 902 (2016) 292.
- [34] D.V. Bykov, A gauged linear formulation for flag-manifold σ -models, Theor. Math. Phys. 193 (3) (2017) 1737, Teor. Mat. Fiz. 193 (3) (2017) 381.
- [35] D.V. Bykov, Cyclic gradings of Lie algebras and Lax pairs for σ -models, Theor. Math. Phys. 189 (3) (2016) 1734, Teor. Mat. Fiz. 189 (3) (2016) 380.
- [36] D. Bykov, Complex structure-induced deformations of σ -models, J. High Energy Phys. 1703 (2017) 130, arXiv:1611.07116 [hep-th].
- [37] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, Unconstrained off-shell $N = 3$ supersymmetric Yang–Mills theory, Class. Quantum Gravity 2 (1985) 155.
- [38] S. Fedoruk, E. Ivanov, Multiparticle $\mathcal{N} = 8$ mechanics with $F(4)$ superconformal symmetry, Nucl. Phys. B 938 (2019) 714, arXiv:1810.13366 [hep-th].
- [39] Z. Zhang, et al., Spectroscopic observation of $SU(N)$ -symmetric interactions in Sr orbital magnetism, Science 345 (6203) (2014) 1467–1473.
- [40] D. Bykov, Haldane limits via Lagrangian embeddings, Nucl. Phys. B 855 (2012) 100, arXiv:1104.1419.
- [41] D. Bykov, The geometry of antiferromagnetic spin chains, Commun. Math. Phys. 322 (2013) 807.
- [42] M. Lajkó, K. Wamer, F. Mila, I. Affleck, Generalization of the Haldane conjecture to $SU(3)$ chains, Nucl. Phys. B 924 (2017) 508.
- [43] Y. Tanizaki, T. Sulejmanpasic, Anomaly and global inconsistency matching: θ -angles, $SU(3)/U(1)^2$ nonlinear sigma model, $SU(3)$ chains and its generalizations, Phys. Rev. B 98 (11) (2018) 115126.
- [44] K. Ohmori, N. Seiberg, S.H. Shao, Sigma models on flags, arXiv:1809.10604 [hep-th].
- [45] R. Donagi, E. Sharpe, GLSM’s for partial flag manifolds, J. Geom. Phys. 58 (2008) 1662.
- [46] V. Ginzburg, Lectures on Nakajima’s quiver varieties, in: Geometric Methods in Representation Theory. I, in: Sémin. Congr., vol. 24-I, Soc. Math. France, Paris, 2012, pp. 145–219.
- [47] A. Borel, F. Hirzebruch, Characteristic classes and homogeneous spaces. I, Am. J. Math. 80 (1958) 458–538.
- [48] S.M. Salamon, F.E. Burstall, Tournaments, flags, and harmonic maps, Math. Ann. 277 (1987) 249–266.
- [49] H. Eichenherr, M. Forger, On the dual symmetry of the nonlinear sigma models, Nucl. Phys. B 155 (1979) 381.
- [50] N. Beisert, F. Luecker, Construction of Lax connections by exponentiation, J. Math. Phys. 53 (2012) 122304.
- [51] S. Salamon, Harmonic and holomorphic maps, in: Geometry Seminar “Luigi Bianchi”, II, 1984, in: Lecture Notes in Math., vol. 1164, Springer, Berlin, 1985, pp. 161–224.
- [52] J.-B. Butruille, Homogeneous nearly Kähler manifolds, arXiv:math/0612655v1, 2006.
- [53] V.G. Kac, Automorphisms of finite order of semisimple Lie algebras, Funkc. Anal. Prilož. 3 (3) (1969) 94–96.
- [54] A. Agricola, A. Borowka, T. Friedrich, S^6 and the geometry of nearly Kähler 6-manifolds, arXiv:1707.08591, 2017.
- [55] I.Ya. Arefeva, Quantum contour field equations, Phys. Lett. B 93 (3) (1980) 347–353.
- [56] I.Ya. Arefeva, Wilson expansion for chiral field, Theor. Math. Phys. 36 (1) (1978) 573–578, Teor. Mat. Fiz. 36 (1) (1978) 24–31.
- [57] S. Sternberg, Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang–Mills field, Proc. Natl. Acad. Sci. 74 (1977) 5253.
- [58] J. Wess, B. Zumino, Consequences of anomalous Ward identities, Phys. Lett. B 37 (1971) 95.
- [59] S.P. Novikov, The Hamiltonian formalism and a many-valued analogue of Morse theory, Russ. Math. Surv. 37 (5) (1982) 1.
- [60] E. Witten, Global aspects of current algebra, Nucl. Phys. B 223 (1983) 422.
- [61] S.K. Wong, Field and particle equations for the classical Yang–Mills field and particles with isotopic spin, Nuovo Cimento A 65 (1970) 689.
- [62] A. Alekseev, L.D. Faddeev, S.L. Shatashvili, Quantization of symplectic orbits of compact Lie groups by means of the functional integral, J. Geom. Phys. 5 (1988) 391.