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The fuzzy 4-hyperboloid H_n^4 and higher-spin in Yang–Mills matrix models

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Abstract

We consider the $SO(4,1)$-covariant fuzzy hyperboloid H_n^4 as a solution of Yang–Mills matrix models, and study the resulting higher-spin gauge theory. The degrees of freedom can be identified with functions on classical H^4 taking values in a higher-spin algebra associated to $\mathfrak{so}(4,1)$, truncated at spin n. We develop a suitable calculus to classify the higher-spin modes, and show that the tangential modes are stable. The metric fluctuations encode one of the spin 2 modes, however they do not propagate in the classical matrix model. Gravity is argued to arise upon taking into account induced gravity terms. This formalism can be applied to the cosmological FLRW space-time solutions of [1], which arise as projections of H_n^4. We establish a one-to-one correspondence between the tangential fluctuations of these spaces.
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1. Introduction

In the present paper we continue the exploration of 4-dimensional covariant fuzzy spaces and their associated higher-spin gauge theories, as started in [2,3]. These are non-commutative spaces which allow to reconcile a quantum structure of space(-time) with covariance under the maximal isometry. In contrast, quantized Poisson manifolds such as $\mathbb{R}^4_\theta$ [4,5] are not fully covariant, as an explicit tensor $\theta^{\mu\nu}$ breaks the covariance. In previous work [2,3], gauge theory on the fuzzy 4-sphere S_N^4 was studied in detail, starting from the observation that S_N^4 is a solution of Yang–

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Mills matrix models supplemented by a mass term, cf. [6]. Here we extend this analysis to fuzzy H^4_n, which is a non-compact quantum space preserving an $SO(4,1)$ isometry, also known as Euclidean AdS^4. For other related work on covariant quantum spaces see e.g. [7–14].

The motivation for this work is two-fold: first, we want to develop a formalism to study gauge theory on H^4_n along the lines of usual calculus and field theory, in order to facilitate the interpretation of the resulting models. While S^4_N allows to use a clean but less intuitive organization of fields into polynomials corresponding to Young diagrams, the non-compact nature of H^4 requires to develop a calculus as well as field formalism reminiscent of the conventional treatment. We will achieve this goal, and obtain results analogous to the compact case but in a more transparent manner.

The second motivation is to set the stage for a similar analysis of the cosmological fuzzy space-time solutions $\mathcal{M}^{3,1}_n$ found in [1,15]. These FLRW-type space-times have very interesting physical properties such as a regularized Big-Bang-like initial singularity and a finite density of microstates. $\mathcal{M}^{3,1}_n$ can obtained from the present H^4_n via a projection, which not only leads to a Minkowski signature, but also reduces the symmetry to $SO(3,1)$. Since the group theory becomes weaker, it seems advisable to consider first the simpler (Euclidean) case of fuzzy H^4_n. We establish the relevant formalism in this paper, and moreover provide some explicit links between the modes on H^4_n and $\mathcal{M}^{3,1}_n$.

One of the most interesting features of 4-dimensional covariant fuzzy spaces is the natural appearance of higher spin theories. This can be understood by recalling that these spaces are quantized equivariant S^2-bundles over the base space (i.e. S^4 or H^4 here), where the fiber is given by the variety of self-dual 2-forms on the base. The equivariant structure implies that would-be Kaluza–Klein modes transmute into higher-spin modes. Taken as background solution in matrix models, such as the IKKT model, one obtains a higher-spin gauge theory as effective theory around the 4-dimensional covariant fuzzy spaces. As a remark, the structure is reminiscent of twistor constructions, see also [16].

Let us describe the results of this paper in some detail. Starting from the classical as well as fuzzy geometry of the hyperboloid H^4, we develop a calculus, solely based on the Poisson structure, to organize the fuzzy algebra of functions on H^4_n into $SO(4,1)$ irreducible components. We further establish a map between the modes in the irreducible components, suggestively called spin s fields, and conventional (rank s) tensor fields on H^4.

Having understood the "functions" on H^4_n, we proceed by considering H^4_n as background in the IKKT matrix model. As a first result, we classify all (tangential) fluctuation modes at a given spin level and exhibit their algebraic features. Subsequently, we are able to diagonalize the kinetic term in the action governing the fluctuations. Remarkably, the kinetic terms for all tangential fluctuations are non-negative such that no instabilities in the tangential sector exist.

Having in mind emergent gravity scenarios, we derive the associated graviton modes for spin 0, 1 and 2 fields. The spin 0 and spin 2 contributions satisfy the de Donder gauge, and at spin 2 one graviton mode emerges from the tangential sector. However, while the underlying modes do propagate, the graviton turns out to behave like an auxiliary field, and does not propagate at the classical level. The reason is that the field redefinition required for the graviton cancels the propagator, similar as in on S^4_N [2].

Nevertheless, our results are interesting and useful. First of all, since classical GR is not renormalizable, it should presumably be viewed as a low-energy effective theory. Then the starting point of an underlying quantum theory should be quite different from GR at the classical level, as in our approach, and gravity may be induced by quantum effects [17,18]. This is the idea of emergent gravity. The present model may well realize this idea, since the basic framework is non-perturbative and well suited for quantization (in particular the maximally supersymmetric IKKT model), and the required spin 2 fluctuations do arise naturally. The extra degrees of freedom may or may not help, but certainly covariance provides a significant advantage compared to other related frameworks, cf. [19]. In particular, it is remarkable that no negative or ghost-like modes appear in the tangential modes.

Perhaps the most interesting perspective is the extension to the cosmological space-times $\mathcal{M}^{3,1}$. We will establish a one-to-one correspondence of the tangential modes on H^4_n to the full set of fluctuations on $\mathcal{M}^{3,1}$. Since the tangential modes on H^4_n are stable and free of pathologies (in contrast to off-shell GR), it seems likely that the Minkowski setting on $\mathcal{M}^{3,1}$ provides a good model, too. In fact, the presence of negative radial modes on H^4_n would require to implement a constraint in the matrix model, which may spoil supersymmetry. This is not needed for $\mathcal{M}^{3,1}$, which provides further motivation for including a discussion of $\mathcal{M}^{3,1}$ here. However, to keep the paper within bounds, we postpone the details for this case to future work.

The paper is organized as follows: We start with a discussion of the classical geometry underlying H^4_n in section 2, before discussing fuzzy H^4_n in detail in section 3. In particular, we introduce a calculus suitable for decomposing the algebra of functions into modules of equal spin. The details of the decomposition and the properties of the irreducible modes are provided in section 4. Having established the fundamentals of fuzzy H^4_n, we explore the fluctuations around an H^4_n background in the IKKT matrix model in section 5. We pay particular attention to the classification of tangential fluctuations, and explicitly diagonalize their kinetic term. Subsequently, the graviton modes are identified and their equation of motions are derived. Before concluding we briefly explore the projection of H^4_n to the Minkowskian $\mathcal{M}^{3,1}_n$ in section 6. Finally, section 7 concludes and provides an outlook for future work. Relevant notation and conventions as well as auxiliary identities and derivations are collected in appendices A–D.

2. Classical geometry underlying H^4_n

The classical geometry underlying fuzzy H^4_n is $\mathbb{C}P^{1,2}$, which is an S^2-bundle over the 4-hyperboloid H^4. More precisely, $\mathbb{C}P^{1,2}$ is an $SO(4,1)$-equivariant bundle over H^4 as well as a coadjoint orbit of $SO(4,2)$. Recall, for instance from [20, Def. 1.5], that a G-equivariant bundle $\pi : E \to X$ is equipped with a G-group action $\tilde{\rho} : E \to E$ as well as $\rho : X \to X$ such that the projection map π is an intertwiner, i.e. $\pi \circ \tilde{\rho} = \rho \circ \pi$. Here, the actions of $SO(4,1)$ on the total space $\mathbb{C}P^{1,2}$ and base space H^4 are immanent by definition of these spaces. In particular, this means that the local stabilizer group $SO(4)$ acts non-trivially on the fiber S^2, leading to higher-spin fields on H^4, and a canonical quantization exists. The construction is similar to twistor constructions for Minkowski space.

2.1. $\mathbb{C}P^{1,2}$ as $SO(4,1)$-equivariant bundle over the hyperboloid H^4

Let $\psi \in \mathbb{C}^4$ be a spinor of $\mathfrak{so}(4,1)$ with $\bar{\psi}\psi = 1$. Consider the following Hopf map:

$$\begin{aligned} H^{4,3} &\to H^4 \subset \mathbb{R}^{1,4} \\ \psi &\mapsto x^a = \frac{r}{2}\bar{\psi}\gamma^a\psi, \qquad a = 0,1,2,3,4\,, \end{aligned} \tag{2.1}$$

where r introduces a length scale, and $H^{4,3}$ is the 7-hyperboloid

$$H^{4,3} = \{\psi \in \mathbb{C}^4 | \, \bar{\psi}\psi = \psi^\dagger \gamma^0 \psi = 1\} \, . \tag{2.2}$$

The γ^a, $a = 0, \ldots, 4$ are $SO(4,1)$ gamma matrices, see appendix B for details. The map (2.1) is a non-compact version of the Hopf map $S^7 \to S^4$, which respects $SO(4,1)$ and in which the x^a transform as $SO(4,1)$ vectors. By using (B.3) one can verify that

$$\sum_{a,b=0}^{4} \eta_{ab} x^a x^b = -\frac{r^2}{4} =: -R^2 \tag{2.3}$$

so that the right-hand side is indeed in H^4; note that $x_a \in \mathbb{R}$ due to (A.7). Since the overall phase of ψ drops out, we can re-interpret (2.1) as a map

$$x^a : \quad \mathbb{C}P^{1,2} \to H^4 \subset \mathbb{R}^{1,4} \tag{2.4}$$

where $\mathbb{C}P^{1,2} = H^{4,3}/U(1)$ is defined as space of unit spinors $\bar{\psi}\psi = 1$ modulo $U(1)$. In other words, $\mathbb{C}P^{1,2}$ is a S^2-bundle over H^4. To exhibit the fiber, consider an arbitrary spinor ψ with $\bar{\psi}\psi = 1$. Since

$$x^0 = \frac{r}{2}\psi^\dagger\psi \; > 0, \tag{2.5}$$

there exists a suitable $SO(4,1)$ transformation such that

$$x^a|_\xi = R(1,0,0,0,0), \tag{2.6}$$

which defines a reference point $\xi \in H^4$. Its stabilizer group is

$$H = \{h; [h, \gamma_0] = 0\} \; = SU(2)_R \times SU(2)_L \subset SO(4,1) \tag{2.7}$$

where $SU(2)_L$ acts on the $+1$ eigenspace of γ^0. By introducing complex parameters

$$\psi^T = \left(a_1^*, a_2^*, b_1, b_2\right), \qquad 1 = \bar{\psi}\psi = -|a_1|^2 - |a_2|^2 + |b_1|^2 + |b_2|^2 = \psi^\dagger\psi \tag{2.8}$$

it follows that $|b_1|^2 + |b_2|^2 = 1$ and $a_1 = a_2 = 0$. Thus after an appropriate $SU(2)_L$ transformation we can assume

$$\psi^T = (0,0,0,1) \, , \tag{2.9}$$

which will be a reference spinor over ξ throughout the remainder. Hence $\mathbb{C}P^{1,2}$ is a S^2-bundle over H^4, and the S^2 fiber is obtained by acting with $SU(2)_L$ on ψ. This is analogous to the well-known fact that $\mathbb{C}P^3$ is an S^2-bundle over S^4. Note that the metric on the hyperboloid induced via

$$x^a : \quad H^4 \hookrightarrow \mathbb{R}^{1,4} \tag{2.10}$$

is Euclidean, despite the $SO(4,1)$ metric on target space. This is obvious at the point $\xi = (R,0,0,0,0)$, where the tangent space is $\mathbb{R}^4_{1234}$.

$SO(4,2)$ formulation and embedding functions. It is useful to view $\mathbb{C}P^{1,2}$ as a 6-dimensional coadjoint orbit of $SU(2,2)$

$$\mathbb{C}P^{1,2} \cong \{U^{-1}ZU, \quad U \in SU(2,2)\} \; \hookrightarrow \; \mathfrak{su}(2,2) \tag{2.11}$$

through the rank one 4×4 matrix

$$Z = \psi\bar{\psi}, \qquad Z^2 = Z, \qquad \mathrm{tr}(Z) = 1, \qquad Z^\dagger = \gamma^0 Z {\gamma^0}^{-1} . \tag{2.12}$$

The embedding (2.11) is described by the embedding functions

$$\begin{aligned} m^{ab} &= \mathrm{tr}(Z\Sigma^{ab}) = \overline{\psi}\Sigma^{ab}\psi = (m^{ab})^*, \\ x^a &= r\,\mathrm{tr}(Z\Sigma^{a5}) = \frac{r}{2}\overline{\psi}\gamma^a\psi = r\,m^{a5}, \qquad a,b = 0,\ldots,4 \end{aligned} \tag{2.13}$$

noting that $\frac{1}{2}\gamma^a = \Sigma^{a5}$, see (A.5). Upon restricting to $\mathfrak{so}(4,1) \subset \mathfrak{so}(4,2) \cong \mathfrak{su}(2,2)$, we recover (2.4), which reflects that the $SO(4,1)$ action is transitive on $\mathbb{C}P^{1,2}$. The last equation in (2.13) amounts to a group-theoretical definition of the Hopf map, which will generalize to the non-commutative case. The $SO(4,2)$ structure is often useful, but it does not respect the projection to H^4.

We can compute the invariant functions

$$\sum_{0\le a<b\le 4} m^{ab}m_{ab} = \sum_{0\le a<b\le 4} \bar{\psi}\bar{\psi}\Sigma^{ab}\otimes\Sigma_{ab}\psi\psi = \frac{1}{2}, \tag{2.14}$$

$$\sum_{0\le a<b\le 5} m^{ab}m_{ab} = \sum_{0\le a<b\le 5} \bar{\psi}\bar{\psi}\Sigma^{ab}\otimes\Sigma_{ab}\psi\psi = \frac{3}{4}, \tag{2.15}$$

using the identities (B.6) and (B.7). Here, the indices are raised and lowered with $\eta_{ab} = \mathrm{diag}(-1,1,1,1,1,-1)$. Combining the two identities (B.6)–(B.7) and recalling $x^a = rm^{a5}$, we recover

$$x_a x^a = -\frac{r^2}{4} = -R^2. \tag{2.16}$$

Remarkably, the $SO(4,1)$-invariant $x^a x_a$ is constant on $\mathbb{C}P^{1,2}$. Similarly, (B.9) together with the above relations imply[1] the $SO(4,2)$ identities

$$\eta_{cc'}m^{ac}m^{bc'} = \frac{1}{4}\eta^{ab}, \qquad a,b = 0,\ldots,5 \tag{2.17}$$

which reduces to the $SO(4,1)$ relation

$$\eta_{cc'}m^{ac}m^{bc'} - r^{-2}x^a x^b = \frac{1}{4}\eta_{ab}, \qquad a,b = 0,\ldots,4 . \tag{2.18}$$

In particular, this implies that m^{ab} is orthogonal to x^a,

$$x_a m^{ab} = 0 . \tag{2.19}$$

Furthermore, the following $SO(4,2)$ identities hold:

$$\begin{aligned} \epsilon_{abcdef}m^{ab}m^{cd} &= \bar{\psi}\bar{\psi}\epsilon_{abcdef}\Sigma^{ab}\otimes\Sigma^{cd}\psi\psi & (2.20)\\ &= 2\bar{\psi}\bar{\psi}(\Sigma^{ef}\otimes 1 + 1\otimes\Sigma^{ef})\psi\psi \\ &= 4\bar{\psi}\Sigma^{ef}\psi = 4m_{ef} , & (2.21) \end{aligned}$$

using (B.10); this can also be seen from (B.11). Reduced to $SO(4,1)$, this implies

[1] This is just a manifestation of the relation $Z^2 = Z$, see (2.12).

$$\epsilon_{abcde} m^{ab} m^{cd} = -\frac{4}{r} x_e \,, \qquad e = 0, \ldots, 4. \tag{2.22}$$

Finally, there exists a self-duality relation

$$\begin{aligned}\epsilon_{abcde} m^{ab} x^c &= \bar\psi \bar\psi \epsilon_{abc5de} \Sigma^{ab} \otimes \Sigma^{c5} \psi \psi &(2.23)\\ &= \frac{1}{2} \bar\psi \bar\psi (\Sigma_{de} \otimes 1 + 1 \otimes \Sigma_{de}) \psi \psi \\ &= \bar\psi \Sigma_{de} \psi = m_{de} &(2.24)\end{aligned}$$

using (B.10). Thus m^{ab} is a tangential self-dual rank 2 tensor on H^4, in complete analogy to S^4_N [21]. At the reference point (2.6), one can express m^{ab} in terms of the $SO(4)$ t'Hooft symbols

$$m^{\mu\nu} = \eta^i_{\mu\nu} J_i \,, \qquad J_i J^i = 1 \tag{2.25}$$

where J_i describes the internal S^2. This exhibits the structure of $\mathbb{C}P^{1,2}$ is an $SO(4,1)$-equivariant bundle over H^4. The fiber S^2 is generated by the local $SU(2)_L$, while $SU(2)_R$ acts trivially.

2.2. $\mathbb{C}P^{1,2}$ as $SO(3,2)$-equivariant bundle over the hyperboloid $H^{2,2}$

Equivalently, the homogeneous space $\mathbb{C}P^{1,2}$ of $SO(4,2)$ can be viewed as S^2-bundle over $H^{2,2}$, which arises from a different Hopf map

$$H^{4,3} \to \mathbb{C}P^{1,2} \to H^{2,2} \subset \mathbb{R}^{2,3} \tag{2.26}$$

as follows, cf. [22]:

$$t^a = \frac{1}{R} \overline{\psi} \Sigma^{a4} \psi = \frac{1}{R} m^{a4}, \qquad a = 0, 1, 2, 3, 5 \,. \tag{2.27}$$

This map is compatible with $SO(3,2)$, and establishes (2.26) as $SO(3,2)$-equivariant bundle in the aforementioned sense. The reference spinor (2.9) is now projected to $t^a = r^{-1}(0,0,0,0,1) \in \mathbb{R}^{3,2}$, which transforms as $SO(3,2)$ vector. Then t^a defines a hyperboloid $H^{2,2} \subset \mathbb{R}^{3,2}$ with intrinsic signature $(+,+,-,-)$. Using analogous identities as before, we obtain the constraints

$$\begin{aligned}&\tilde\eta_{ab} t^a t^b = r^{-2}, \qquad \tilde\eta_{ab} = \mathrm{diag}(-1,1,1,1,-1) \,,\\ &\quad t_a x^a = 0 = t_\mu x^\mu \,.\end{aligned} \tag{2.28}$$

The last relation follows from the $SO(4,2)$ relation (2.17), noting that $t^4 \equiv 0$. More generally, we can consider

$$x^a = m^{ab} \alpha_b, \qquad t^a = m^{ab} \beta_b \tag{2.29}$$

where $\alpha, \beta \in \mathbb{R}^{2,4}$ are two linearly independent vectors with[2] $\alpha_b \beta^b = 0$. Then the previous constructions are recovered for $\alpha = e^5$, $\beta = e^4$. The common symmetry group which preserves both α_b and β_b is $SO(3,1)$. Note that $t^5 \propto x^4$ on $\mathbb{C}P^{1,2}$.

[2] The case of light-like α is also interesting, see section 3.2.

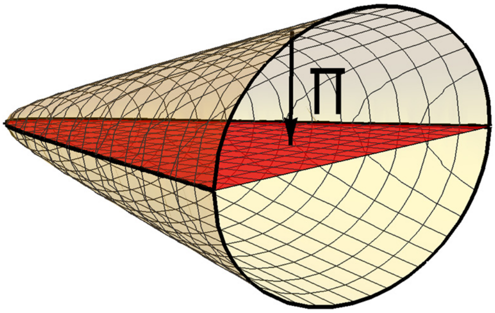

Fig. 1. Sketch of the projection Π_x from H^4 to $\mathcal{M}^{3,1}$ with Minkowski signature.

2.3. $SO(3,1)$-invariant projections and Minkowski signature

So far we have constructed H^4 and $H^{2,2}$, but not a space with Minkowski signature yet. Space-times with Minkowski signature can be obtained by $SO(3,1)$-covariant projections of the above hyperboloids. Explicitly, consider the projections

$$\begin{aligned} \Pi_x: \quad & \mathbb{C}P^{1,2} \to \mathbb{R}^{3,1}, \\ & m \mapsto x^\mu = m^{\mu b}\alpha_b \\ \Pi_t: \quad & \mathbb{C}P^{1,2} \to \mathbb{R}^{3,1}, \\ & m \mapsto t^\mu = m^{\mu b}\beta_b \end{aligned} \qquad \text{with } \mu = 0,1,2,3, \tag{2.30}$$

which respect $SO(3,1)$. A sketch of Π_x is displayed in Fig. 1. In section 6, the image $\mathcal{M}^{3,1} \subset \mathbb{R}^{3,1}$ of Π_x serves as cosmological FLRW space-time with $k = -1$, as discussed in [1]. In contrast, t^μ is interpreted as internal space related to translations.

3. The fuzzy hyperboloid H^4_n

Now we turn to the central object of this paper: the fuzzy hyperboloid H^4_n. H^4_n is a quantization of the bundle $\mathbb{C}P^{1,2}$ over H^4, which respects the $SO(4,2)$ structure and the projection to the base space H^4. This is natural because $\mathbb{C}P^{1,2}$ is a coadjoint orbit of $SO(4,2)$ via (2.11). As such $\mathbb{C}P^{1,2}$ is equipped with a canonical $SO(4,2)$-invariant Poisson (symplectic) structure; whereas on H^4 no such structure exists. H^4_n was first discussed in [22], and it serves as starting point for a quantized cosmological space-time in [15].

As for any coadjoint orbit, fuzzy H^4_n can be defined in terms of the operator algebra $\mathrm{End}(\mathcal{H}_n)$, where $\mathcal{H}_n$ is a suitable unitary irrep of $SU(2,2) \cong SO(4,2)$. The representation is chosen such that the Lie algebra generators $\mathcal{M}_{ab} \in \mathrm{End}(\mathcal{H}_n)$ generate a non-commutative algebra of functions, interpreted as quantized or fuzzy $\mathbb{C}P^{1,2}_n$. The $\mathcal{M}^{ab}$ are naturally viewed as quantized coordinate functions m^{ab} (2.13) on $\mathbb{C}P^{1,2}$. Fuzzy H^4_n is then generated by Hermitian generators $X^a \sim x^a$, which transform as vectors under $SO(4,1) \subset SO(4,2)$, and are interpreted as quantized embedding functions (2.4). This will be made more explicit through an oscillator construction, which allows to derive all the required properties.

To define fuzzy H^4_n explicitly, let $\eta^{ab} = \mathrm{diag}(-1,1,1,1,1,-1)$ be the invariant metric of $SO(4,2)$, and let $\mathcal{M}^{ab}$ be the Hermitian generators of $SO(4,2)$, which satisfy

$$[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = \mathrm{i}(\eta_{ac}\mathcal{M}_{bd} - \eta_{ad}\mathcal{M}_{bc} - \eta_{bc}\mathcal{M}_{ad} + \eta_{bd}\mathcal{M}_{ac}) \,. \tag{3.1}$$

We choose a particular type of (discrete series) positive-energy unitary irreps[3] $\mathcal{H}_n$ known as *minireps* or *doubletons* [23,24]. Remarkably, the $\mathcal{H}_n$ remain irreducible[4] under $SO(4,1) \subset SO(4,2)$. Moreover, the minireps have positive discrete spectrum

$$\mathrm{spec}(\mathcal{M}^{05}) = \{E_0, E_0+1, \dots\}, \qquad E_0 = 1 + \frac{n}{2} \tag{3.2}$$

where the eigenspace with lowest eigenvalue of $\mathcal{M}^{05}$ is an $n+1$-dimensional irreducible representation of either $SU(2)_L$ or $SU(2)_R$. Then the Hermitian generators

$$\begin{aligned} X^a &:= r\mathcal{M}^{a5}, \qquad a = 0,\dots,4 \\ [X^a, X^b] &= -\mathrm{i}r^2\mathcal{M}^{ab} =: \mathrm{i}\Theta^{ab} \end{aligned} \tag{3.3}$$

(note the signs!) transform as $SO(4,1)$ vectors, i.e.

$$\boxed{\begin{aligned} [\mathcal{M}_{ab}, X_c] &= \mathrm{i}(\eta_{ac}X_b - \eta_{bk}X_a), \\ [\mathcal{M}_{ab}, \mathcal{M}_{cd}] &= \mathrm{i}(\eta_{ac}\mathcal{M}_{bd} - \eta_{ad}\mathcal{M}_{bc} - \eta_{bc}\mathcal{M}_{ad} + \eta_{bd}\mathcal{M}_{ac}) \,. \end{aligned}} \tag{3.4}$$

Because the restriction to $SO(4,1) \subset SO(4,2)$ is irreducible, it follows that the X^a live on a hyperboloid,

$$\boxed{\eta_{ab}X^aX^b = X^iX^i - X^0X^0 =: -R^2\mathbb{1}} \tag{3.5}$$

with some R^2 to be determined below. Since $X^0 = r\mathcal{M}^{05} > 0$ has positive spectrum, this describes a one-sided hyperboloid in $\mathbb{R}^{1,4}$, denoted as H^4_n. Analogous to fuzzy S^4_N, the semi-classical geometry underlying H^4_n is $\mathbb{C}P^{1,2}$ [22], which is an S^2-bundle over H^4 carrying a canonical symplectic structure. In the fuzzy case, this fiber is a fuzzy 2-sphere S^2_n. We work again in the semi-classical limit. We also note the following commutation relations

$$\Box_X X^b = [X_a, [X^a, X^b]] = -4r^2X^b \,. \tag{3.6}$$

The negative sign arises from $\eta = \mathrm{diag}(-1,1,1,1,1,-1)$, and $\Box_X$ is not positive definite.

3.1. Fuzzy $H^{2,2}_n$ and momentum space

As in the classical case (2.27) and for later purpose, we also define

$$T^a = \frac{1}{R}\mathcal{M}^{a4}, \qquad a = 0,\dots,3,5 \tag{3.7}$$

where $RrT^5 = -X^4$. As the restriction of $\mathcal{H}_n$ to $SO(3,2) \subset SO(4,2)$ is irreducible, the operators (3.7) satisfy the constraint

$$\tilde{\eta}_{ab}T^aT^b = -T^0T^0 + \sum_{i=1,2,3} T^iT^i - T^5T^5 = \frac{1}{r^2}\mathbb{1} \tag{3.8}$$

[3] Strictly speaking there are two versions $\mathcal{H}^L_n$ or $\mathcal{H}^R_n$ with opposite “chirality”, but this distinction is irrelevant in the present paper and therefore dropped.

[4] This follows from the minimal oscillator construction of $\mathcal{H}_n$, where all $SO(4,2)$ weight multiplicities are at most one, cf. [23,25,26].

cf. (2.28). This is the quantization of the hyperboloid $H^{2,2} \subset \mathbb{R}^{3,2}$ with intrinsic signature $(+,+,-,-)$ of section 2.2 and becomes Lorentzian via the projection (2.3). The commutation relations are

$$\begin{aligned}[][T^a, T^b] &= \mathrm{i}\frac{1}{R^2}\mathcal{M}^{ab} \qquad a,b = 0,\ldots,3,5\,,\\ [T^\mu, X^\nu] &= \mathrm{i}\frac{1}{R}\eta^{\mu\nu}X^4, \qquad \mu,\nu = 0,\ldots,3\,,\end{aligned} \tag{3.9}$$

which justifies to consider T^μ as translation generators, and

$$\Box_T T^b = [T_a, [T^a, T^b]] = +\frac{4}{R^2}T^b \tag{3.10}$$

Note the different signs in (3.10) and (3.6), which arise from of $\eta^{55} = -1 = -\eta^{44}$.

3.2. $SO(3,1)$-covariant fuzzy spaces

In analogy to section 2.3, we consider the $SO(3,1)$-covariant fuzzy generators

$$\tilde{X}^\mu = \mathcal{M}^{\mu a}\alpha_a, \qquad \tilde{T}_\mu = \mathcal{M}^{\mu a}\beta_a \tag{3.11}$$

where α, β are $SO(3,1)$-invariant. They satisfy

$$\begin{aligned}[][\tilde{X}^\mu, \tilde{X}^\nu] &= (\alpha\cdot\alpha)\mathcal{M}^{\mu\nu}, \qquad [\tilde{T}^\mu, \tilde{T}^\nu] = (\beta\cdot\beta)\mathcal{M}^{\mu\nu}\\ [\tilde{X}^\mu, \tilde{T}_\nu] &= \mathrm{i}\big(\delta^\mu_\nu \mathcal{M}^{ab}\alpha_a\beta_b + \alpha\cdot\beta\mathcal{M}^{\mu\nu}\big)\\ &= \mathrm{i}\big(\delta^\mu_\nu\, \alpha\wedge\beta D + \alpha\cdot\beta\mathcal{M}^{\mu\nu}\big)\end{aligned} \tag{3.12}$$

where $\alpha\wedge\beta = \alpha^4\beta^5 - \alpha^5\beta^4$ and $D = \mathcal{M}^{45}$. For $\alpha\cdot\alpha \approx 0$ and $\alpha\cdot\beta \approx 1 \approx \alpha\wedge\beta$, the $\tilde{X}^\mu$ become almost commutative and the commutation relations are not far from the Poincare algebra:

Poincare algebra. In particular for light-like $\alpha = \frac{1}{\sqrt{2}}(1,-1)$ and $\beta = \frac{1}{\sqrt{2}}(1,1)$, we obtain

$$K_\mu := \frac{1}{\sqrt{2}}(\mathcal{M}_{\mu 5} - \mathcal{M}_{\mu 4}), \qquad \tilde{T}^\mu = \frac{1}{\sqrt{2}}(\mathcal{M}^{\mu 5} + \mathcal{M}^{\mu 4}) \tag{3.13}$$

which satisfy

$$\begin{aligned}[][\tilde{T}^\mu, \tilde{T}^\nu] &= 0 = [K^\mu, K^\nu]\,,\\ [\tilde{T}^\mu, K_\nu] &= \mathrm{i}(\delta^\mu_\nu D + \mathcal{M}^{\mu\nu})\,.\end{aligned} \tag{3.14}$$

Hence the $\tilde{T}^\mu$ together with $\mathcal{M}^{\mu\nu}$ generate the Poincare algebra $ISO(3,1)$ as sub-algebra of $\mathfrak{so}(4,2)$, with special conformal generators K^μ and the dilatation operator D

$$[D, \tilde{T}_\mu] = \mathrm{i}\tilde{T}_\mu, \quad [D, K_\mu] = -\mathrm{i}K_\mu. \tag{3.15}$$

3.3. Oscillator realization, minireps and coherent states

The Hilbert space $\mathcal{H}_n$ is a highest-weight unitary representation of $SU(2,2)$, which can be obtained by quantizing the spinorial construction of $\mathbb{C}P^{2,1}$ in (2.1). For the quantization one replaces the classical 4-component spinor ψ_α by 4 operators, which satisfy

$$[\psi_\alpha, \bar{\psi}^\beta] = \delta^\beta_\alpha\,. \tag{3.16}$$

The associated bilinears

$$\mathcal{M}^{ab} := \bar{\psi}\,\Sigma^{ab}\psi \tag{3.17}$$

realize the Lie algebra (3.1) of $SO(4,2)$, due to

$$\left[\bar{\psi}\,\Sigma^{ab}\psi, \bar{\psi}\,\Sigma^{cd}\psi\right] = \bar{\psi}\left[\Sigma^{ab}, \Sigma^{cd}\right]\psi. \tag{3.18}$$

The $\mathcal{M}_{ab}$ are self-adjoint operators, since

$$\Sigma^{ab\dagger} = \gamma^0\Sigma^{ab}\gamma^{0-1}\,. \tag{3.19}$$

As a consequence, they implement unitary representations of $SU(2,2)$ on the Fock space $\mathcal{F} = \mathrm{span}\{\bar{\psi}\ldots\bar{\psi}|0\rangle\}$ of the bosonic oscillators, which decomposes into an infinite number of irreducible positive energy unitary representations $\mathcal{H}_\Lambda$.

The oscillator algebra (3.16) can be realized explicitly as follows (cf. [24,27]): Consider bosonic creation and annihilation operators a_i, b_j which satisfy

$$[a_i, a_j^\dagger] = \delta_i^j\,, \qquad [b_i, b_j^\dagger] = \delta_i^j \qquad \text{for } i,j = 1,2\,. \tag{3.20}$$

Using the a_i, b_j we form spinorial operators

$$\psi := \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ b_1 \\ b_2 \end{pmatrix} \tag{3.21}$$

with Dirac conjugates

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 = \left(-a_1, -a_2, b_1^\dagger, b_2^\dagger\right). \tag{3.22}$$

Then

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta_\beta^\alpha \tag{3.23}$$

as required, and the $SO(4,2)$ generators are

$$\mathcal{M}^{ab} = \bar{\psi}\,\Sigma^{ab}\psi = \left(-a_1, -a_2, b_1^\dagger, b_2^\dagger\right)\Sigma^{ab}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ b_1 \\ b_2 \end{pmatrix}\,. \tag{3.24}$$

The generators of $SU(2)_L$ and $SU(2)_R$ are defined by

$$L_i^k := a_k^\dagger a_i - \frac{1}{2}\delta_i^k N_a$$
$$R_j^i := b_i^\dagger b_j - \frac{1}{2}\delta_j^i N_b$$

and the time-like generator X^0 (or the "conformal Hamiltonian" E) is given by

$$r^{-1}X^0 = E = M^{05} = \bar{\psi}\,\Sigma^{05}\psi = \frac{1}{2}\psi^\dagger\psi = \frac{1}{2}(N_a + N_b + 2), \tag{3.25}$$

where $N_a \equiv a_i^\dagger a_i$, $N_b \equiv b_j^\dagger b_j$ are the bosonic number operators, and

$$\hat{N} = \bar{\psi}\psi = -N_a + N_b - 2 \tag{3.26}$$

is invariant. The non-compact generators are given by linear combinations of creation and annihilation operators of the form $a_i^\dagger b_j^\dagger$ and $a_i b_j$.

Minireps. The simplest class of unitary representation has lowest weight space given by the Fock vacuum $a_i\,|0\rangle = 0 = b_i\,|0\rangle$, which defines [27]

$$|\Omega\rangle := |1,0,0\rangle =: |0\rangle\,, \qquad E = 1,\, j_L = j_R = 0\,. \tag{3.27}$$

This gives the *doubleton* minireps built on the lowest weight vectors

$$\begin{aligned} |\Omega\rangle &:= \left|E, \frac{n}{2}, 0\right\rangle := a_{i_1}^\dagger \ldots a_{i_n}^\dagger\,|0\rangle\,, \qquad E = 1 + \frac{n}{2},\, j_L = \frac{n}{2},\;\; j_R = 0 \\ |\Omega\rangle &:= \left|E, 0, \frac{n}{2}\right\rangle := b_{i_1}^\dagger \ldots b_{i_n}^\dagger\,|0\rangle\,, \qquad E = 1 + \frac{n}{2},\, j_L = 0,\, j_R = \frac{n}{2} \end{aligned} \tag{3.28}$$

which are annihilated by all L^- operators, i.e. of the form $a_i b_j$,

$$a_i b_j |\Omega\rangle \equiv 0 \tag{3.29}$$

and

$$n^2 := \left(\hat{N} + 2\right)^2 = (N_a - N_b)^2\,, \qquad n = 0, 1, 2, \ldots\,. \tag{3.30}$$

Acting with all operators of the form $a_i^\dagger b_j^\dagger$ of L^+ on $|\Omega\rangle$, one obtains positive energy discrete series UIR's $\mathcal{H}_\Lambda$ of $U(2,2)$ with lowest weight $\Lambda = \left(E, \frac{n}{2}, 0\right)$ and $\Lambda = \left(E, 0, \frac{n}{2}\right)$. We will largely ignore the distinction and denote both as $\mathcal{H}_n$. These are known as *minireps* of $\mathfrak{so}(4,2)$, because they are free of multiplicities in weight space.[5] They correspond to fields living on the boundary of AdS^5. The minireps remain irreducible under $SO(4,1)$ as well as $SO(3,2)$, and they can be interpreted as massless fields on AdS^4, or as conformal fields[6] on Minkowski space. The lowest weight state $\left|E, 0, \frac{n}{2}\right\rangle$ of $\mathcal{H}_n$ generates a $(n+1)$-dimensional irreducible representation of either $SU(2)_L$ or $SU(2)_R$ with degenerate X^0, naturally interpreted as fuzzy S_n^2.

Comparing the above oscillator construction (3.17) with (2.13), it is manifest that for each $\mathcal{H}_n$, with $n > 0$, the $\mathcal{M}_{ab}$ generators can be interpreted as quantized embedding functions

$$\mathcal{M}_{ab} \sim m_{ab} : \quad \mathbb{C}P^{1,2} \to \mathfrak{so}(4,2) \cong \mathbb{R}^{15}\,. \tag{3.31}$$

This provides the quantization of the coadjoint orbits (2.11), which defines fuzzy $\mathbb{C}P_n^{1,2}$. Since $X^0 \geq 1$, they should be viewed as quantized bundles with base space H_n^4 described by X^a, and fiber S_n^2, for $n = 1, 2, 3, \ldots$. The implicit constraints defining these varieties will be elaborated below. For $n > 0$, these spaces have been briefly discussed in [16,22], and we will mostly focus on that case. The minimal $n = 0$ case is different, but also very interesting, and we discuss it in some detail in appendix C.2.

Coherent states and quantization. The above discrete series irreps $\mathcal{H}_n$ provide a natural definition of coherent states $|m\rangle = g \cdot |\Omega\rangle \in \mathcal{H}_n$, which are given by the $SO(4,2)$ orbit through the lowest

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5 This can be seen e.g. from the characters given in [26].

6 It may seem tempting to apply some of the standard technology of CFT in the present context. However, the use of $SO(4,2)$ here is quite different from CFT, and it does not respect the bundle structure over H^4. Also, the notions of primaries and descendants do not seem to be applicable here, since in the present signature K^μ (3.15) do not rise or lower the eigenvalues of D.

weight state $|\Omega\rangle$. The set of coherent states forms a $U(1)$-bundle over $\mathbb{C}P^{1,2}$, and allow to recover the semi-classical geometry of $\mathbb{C}P^2$ as S^2-bundle over H^4 via $m^{ab} = \langle m| \mathcal{M}^{ab} |m\rangle$. In particular, the lowest weight state is located at the reference point $\langle\Omega| X^a |\Omega\rangle = x^a_\xi = (R, 0, 0, 0, 0)$, see (2.6). The local $SO(4)$ generators $\mathcal{M}^{ij}$ act on the coherent states over ξ in a spin $\frac{n}{2}$ irrep.

These coherent states $|m\rangle$ also provide a $SO(4,2)$-equivariant quantization map from the classical space of functions on $\mathbb{C}P^{1,2}$ to the fuzzy functions $\mathrm{End}(\mathcal{H}_n)$:

$$\begin{aligned} \mathcal{Q}: \quad & \mathcal{C}(\mathbb{C}P^{1,2}) \to \mathrm{End}(\mathcal{H}_n) \\ & f(m) \mapsto \int\limits_{\mathbb{C}P^{1,2}} d\mu\, f(m)\, |m\rangle \langle m| \end{aligned} \tag{3.32}$$

where $|m\rangle$ is a coherent state,[7] and $d\mu$ is the $SO(4,2)$-invariant measure. For polynomial functions, this corresponds to Weyl quantization, mapping irreducible polynomials $P(m^{ab})$ to the corresponding totally symmetrized polynomials $P(\mathcal{M}^{ab})$; in particular $\mathcal{Q}(m^{ab}) = \mathcal{M}^{ab}$. Likewise, square-integrable functions on $\mathbb{C}P^{1,2}$ are mapped to Hilbert-Schmidt operators in $\mathrm{End}(\mathcal{H}_n)$. We expect[8] that the map $\mathcal{Q}$ is surjective, and that all "reasonable" (e.g. square-integrable or Hilbert-Schmidt) harmonics in $\mathrm{End}(\mathcal{H}_n)$ can be obtained as quantizations of higher-spin harmonics on H^4 via $\mathcal{Q}$. This will be used below.

3.4. Algebraic properties of fuzzy H^4_n

Using the aforementioned oscillator realization, one can derive a number of useful identities for the above operators on $\mathcal{H}_n$; we refer the reader to appendix C for the details. To begin with, consider the $SO(4,1)$-invariant radius operator

$$\mathcal{R}^2 := \sum_{a,b=0,1,2,3,4} \eta_{ab} X^a X^b \,. \tag{3.33}$$

Since $\mathcal{H}_\Lambda$ is irreducible under $\mathfrak{so}(4,1)$, it must follow that $\mathcal{R}^2 \sim \mathbb{1}$. Indeed, one finds

$$X_a X^a = -\frac{r^2}{4}\hat{N}(\hat{N}+4) = -\frac{r^2}{4}(n^2-4) =: -R^2 \tag{3.34}$$

where $n = |\hat{N}+2| = 0, 1, 2, \ldots$. Note that $\mathcal{R}^2$ is positive for $n = 0, 1$, which seems strange because X^0 is positive. However, this is a quantum artifact, and the expectation values $\langle X^a\rangle$ under coherent states still sweep out the usual H^4. Additionally we compute the quadratic $SO(4,1)$ and $SO(4,2)$ Casimir operators

$$C^2[\mathfrak{so}(4,1)] = \sum_{a<b\le 4} \mathcal{M}_{ab}\mathcal{M}^{ab} = \frac{1}{2}(n^2-4)\,, \tag{3.35}$$

$$C^2[\mathfrak{so}(4,2)] = \sum_{a<b\le 5} \mathcal{M}_{ab}\mathcal{M}^{ab} = \frac{3}{4}(n^2-4)\,. \tag{3.36}$$

We note that (3.35) agrees with [22]. Further identities can be obtained from the $\mathfrak{so}(6)_\mathbb{C}$ identity (B.9), which entails

[7] Observe that the phase ambiguity of the coherent states drops out here.

[8] For a formal argument see appendix C.1. A more rigorous proof would be desirable.

$$\eta_{cc'}\mathcal{M}^{ac}\mathcal{M}^{bc'} + (a \leftrightarrow b) = \frac{1}{2}(n^2-4)\eta_{ab} \,. \tag{3.37}$$

This implies the $\mathfrak{so}(4,1)$ relation

$$\eta_{cc'}\Theta^{ac}\Theta^{bc'} + (a \leftrightarrow b) = r^2\left(2R^2\eta_{ab} + \left(X^aX^b + X^bX^a\right)\right) \,. \tag{3.38}$$

These correspond to (2.17), (2.18). Moreover, one finds

$$X_b\mathcal{M}^{ab} + \mathcal{M}^{ab}X_b = 0 \,, \tag{3.39}$$

which means that the $SO(4,1)$ generators $\mathcal{M}^{ab}$ are tangential to H^4_n. Another interesting identity is

$$\begin{aligned} \epsilon_{abcdef}\mathcal{M}^{ab}\mathcal{M}^{cd} &= 4n\mathcal{M}_{ef} \\ \epsilon_{abcde}\mathcal{M}^{ab}\mathcal{M}^{cd} &= 4nr^{-1}X_e \end{aligned} \tag{3.40}$$

cf. (2.21), (2.22). Finally, the self-duality relation (2.24) becomes

$$\epsilon_{abcde}\mathcal{M}^{ab}X^c = nr\mathcal{M}_{de} \,. \tag{3.41}$$

To summarize, we have found counterparts for all relation of the classical geometry in section 2.1, which vindicates the choice of representation $\mathcal{H}_n$.

3.5. *Wave-functions and spin Casimir*

Given a representation $\mathcal{H}_n$ of $SO(4,2)$, the most general "function" in $\mathrm{End}(\mathcal{H}_n)$ can always be expanded as follows

$$\phi = \phi(X) + \phi_{ab}(X)\mathcal{M}^{ab} + \ldots \qquad \in \mathrm{End}(\mathcal{H}_n) \;=: \mathcal{C} \,, \tag{3.42}$$

which transform in the adjoint representation $\mathcal{M}^{ab} \mapsto [\mathcal{M}^{ab}, \cdot]$ of $\mathfrak{so}(4,2)$. The $\phi_{ab}(X)$ will be interpreted as quantized tensor fields on H^4, which transform under $SO(4,1)$. We define an $SO(4,2)$-invariant inner product on $\mathcal{C}$ via

$$\langle\phi,\psi\rangle = \mathrm{tr}_{\mathcal{H}}\left(\phi^\dagger\psi\right) \,. \tag{3.43}$$

For polynomials generated by the X^a, this trace diverges. However this is only an IR-divergence, and we are mainly interested in normalizable fluctuations corresponding to physical scalar fields. Technically speaking, we will be working with Hilbert-Schmidt operators in $\mathrm{End}(\mathcal{H})$. These can be expanded into modes obtained by decomposing $\mathrm{End}(\mathcal{H})$ into unitary representations of the isometry group $SO(4,1)$ of the background. We will see that the expansion (3.42) is truncated at n generators $\mathcal{M}_{ab}$.

Spin Casimir. To proceed, we require a characterization of the above $SO(4,1)$ modes in terms of a Casimir operator which measures spin. One can achieve this by the $SO(4,1)$-invariant

$$\mathcal{S}^2 := C^2[\mathfrak{so}(4,1)] + r^{-2}\Box = \sum_{a<b\le 4}[\mathcal{M}^{ab},[\mathcal{M}_{ab},\cdot]] + r^{-2}[X_a,[X^a,\cdot]] \,, \tag{3.44}$$

which measures the spin along the S^2 fiber. To understand this, we locally decompose $\mathfrak{so}(4,1)$, for example at the reference point (2.6), into $\mathfrak{so}(4)$ generators $\mathcal{M}^{\mu\nu}$ and translation generators $P^\mu = \frac{1}{R}\mathcal{M}^{\mu 0}$. Then $C^2[\mathfrak{so}(4,1)] = -R^2P_\mu P^\mu + C^2[\mathfrak{so}(4)]$, and $R^2P_\mu P^\mu \sim$

$-r^{-2}[X_\mu, [X^\mu, \cdot]]$ if acting on functions $\phi(x)$, cf. (3.69). Therefore $\mathcal{S}^2 \sim C^2[\mathfrak{so}(4)]$ should vanish on scalar functions $\phi(X)$ on H^4, but not on higher-spin functions involving θ^{ab}. We will see that this is indeed the case, and $\mathrm{End}(\mathcal{H}_n)$ contains modes up to spin n as measured by $\mathcal{S}^2$ (3.55). We also observe

$$\begin{aligned} C^2[\mathfrak{so}(4,2)] &= C^2[\mathfrak{so}(4,1)] - r^{-2}\Box = \mathcal{S}^2 - 2r^{-2}\Box \,, \\ C^2[\mathfrak{so}(4,2)] &= 2C^2[\mathfrak{so}(4,1)] - \mathcal{S}^2 \,. \end{aligned} \tag{3.45}$$

Note that $\mathcal{S}^2$, $\Box$, and $C^2[\mathfrak{so}(4,2)]$ commute and can be diagonalized simultaneously.

Higher-spin modes on H^4_n. To determine the spectrum of $\mathcal{S}^2$ for the modes in (3.42) we first prove the following identity for any $f \in \mathcal{C}$:

$$\mathcal{S}^2(\{f, X_a\}_+) = \{\mathcal{S}^2 f, X_a\}_+ \,, \tag{3.46}$$

where $\{\cdot,\cdot\}_+$ denote the anti-commutator. To see this, consider

$$\begin{aligned} \mathcal{S}^2(f X_a) &= (\mathcal{S}^2 f)X_a + [\mathcal{M}^{cd}, f][\mathcal{M}_{cd}, X_a] + 2r^{-2}[X^c, f][X_c, X_a] \\ &= (\mathcal{S}^2 f)X_a + 2i[\mathcal{M}^{ad}, f]X_d - 2i[X^c, f]\mathcal{M}_{ca} \\ &= (\mathcal{S}^2 f)X_a + 2i[\mathcal{M}^{ad}X_d, f] - 2i\mathcal{M}^{ad}[X_d, f] - 2i[X^c, f]\mathcal{M}_{ca} \,. \end{aligned} \tag{3.47}$$

Similarly,

$$\mathcal{S}^2(X_a f) = X_a(\mathcal{S}^2 f) + 2i[X_d\mathcal{M}^{ad}, f] - 2i[X_d, f]\mathcal{M}^{ad} - 2i\mathcal{M}_{ca}[X^c, f] \tag{3.48}$$

and adding them yields (3.46). Next, starting from

$$\Box_X X^a = -4X^a = -C^2[\mathfrak{so}(5)]X^a \tag{3.49}$$

this identity immediately implies

$$\mathcal{S}^2 P_n(X) = 0 \tag{3.50}$$

for totally symmetrized polynomials $P_n(X)$ in X. More generally, we show in appendix C.1 that this holds for any scalar field ϕ on H^4 quantized via coherent states, i.e.

$$\mathcal{S}^2\phi = 0 \qquad \text{for any} \quad \phi = \int\limits_{\mathbb{C}P^{1,2}} \phi(x)|x\rangle\langle x| \,. \tag{3.51}$$

As a next step, we consider the higher-spin fields. Using

$$\begin{aligned} 2C^2[\mathfrak{so}(4,1)]\mathcal{M}_{ab} &= [\mathcal{M}^{cd}, [\mathcal{M}_{cd}, \mathcal{M}_{ab}]] = 12\mathcal{M}_{ab} \\ \Box_X \mathcal{M}_{ab} &= [X^c, [X_c, \mathcal{M}_{ab}]] = -2\mathcal{M}_{ab} \\ \mathcal{S}^2\mathcal{M}_{ab} &= 4\mathcal{M}_{ab} \end{aligned} \tag{3.52}$$

we find

$$\mathcal{S}^2\phi^{(1)} = 4\phi^{(1)} \qquad \text{for any} \quad \phi^{(1)} = \phi_{ab}(x)\mathcal{M}^{ab} \tag{3.53}$$

with quantized functions $\phi_{ab}(X)$ on H^4 in (3.51), etc. We can similarly compute $\mathcal{S}^2$ for any irreducible polynomial function in $\mathcal{M}^{ab}$, and obtain

$$\mathcal{S}^2(\Xi^s_{\underline{\alpha}}) = 2s(s+1)\Xi^s_{\underline{\alpha}}, \qquad \Xi^s_{\underline{\alpha}} = (\mathcal{P}_{\underline{\alpha}})_{a_1b_1\ldots a_sb_s}\mathcal{M}^{a_1b_1}\ldots\mathcal{M}^{a_sb_s} \tag{3.54}$$

where $\mathcal{P}_{\underline{\alpha}} \sim$ ⊞⊞ is a 2-row rectangular Young projector. The restriction to these Young diagrams follows from the commutation relations (3.1) and the self-duality relation (3.40). This leads to the decomposition

$$\boxed{\mathcal{C} := \mathrm{End}(\mathcal{H}_n) = \bigoplus_{s=0}^{n} \mathcal{C}^s, \qquad \mathcal{S}^2|_{\mathcal{C}^s} = 2s(s+1)} \tag{3.55}$$

where $\mathcal{C}^s$ is the eigenspace of $\mathcal{S}^2 = 2s(s+1)$. We refer to appendix C.1 for the details. Of course the $\mathcal{C}^s$ contain also forms of the type $\phi_{\underline{\alpha}}(X)\,\Xi^s_{\underline{\alpha}}$. However since the multiplication does not respect the grading, we can only say that $\mathcal{C}^s$ is the *quantization* of tensor fields $\phi_{\underline{\alpha}}(x)$ taking values in the vector space spanned by $\Xi^{\underline{\alpha}}$,

$$\mathcal{C}^s \ \ni\ \mathcal{Q}(\phi_{a_1\ldots a_s;b_1\ldots b_s}(x)m^{a_1b_1}\ldots m^{a_sb_s}) \equiv \mathcal{Q}(\phi_{\underline{\alpha}}(x)\,\Xi^{\underline{\alpha}}), \qquad s \leq n\,, \tag{3.56}$$

where $\Xi^{\underline{\alpha}}$ denotes both the polynomials in $\mathcal{M}^{ab}$ and m^{ab}. We remind the reader that $\mathcal{Q}$ (3.32) respects $\mathfrak{so}(4,2)$. The truncation[9] at n follows provided $\mathcal{Q}$ is surjective, since the corresponding classical expressions (3.56) with $s > n$ are annihilated by $\mathcal{Q}$. In fact they correspond to spin $s > n$ irreps of the local $SO(4)$, which are not supported by the local fiber spanned by the coherent states, which is a fuzzy 2-sphere S^2_n. See appendix C.1 for more details.

This is a very remarkable structure, which leads to higher-spin fields on H^4 truncated at spin n. For small n, the uncertainty scale $L^2_{\mathrm{NC}} \approx R^2$, see (3.59), is set by the curvature scale of $H^4 \subset \mathbb{R}^{1,4}$, so that the space is far from classical. Nevertheless, the case of small n may be interesting after projection to the cosmological space-time $\mathcal{M}^{3,1}$ as discussed in section 6.

The $n = 2$ case. The case $\hat{N} = 0 = n - 2$ is special, because then $C^2[\mathfrak{so}(4,1)] = 0 = R^2$. To avoid this we will assume $n \neq 2$ in this paper.

The $n = 0$ case. In that case, (3.40) gives

$$\epsilon_{abcdef}\mathcal{M}^{ab}\mathcal{M}^{cd} = 0\,, \tag{3.57}$$

which is a relation in the Joseph ideal [28]. Then the $\mathcal{M}^{ab}$, $a, b = 0, \ldots, 5$ generate Vasiliev's higher-spin algebra associated to $\mathfrak{so}(4,2)$. However here we will not aim for a higher-spin theory on AdS^5, but reduce $\mathcal{H}_n$ for $n \neq 0$ to the $\mathfrak{so}(4,1)$ generators $\mathcal{M}^{ab}$, $a, b = 0, \ldots, 4$, and the remaining X^a generators. Then the $\mathcal{M}^{ab}$, $a, b = 0, \ldots, 4$ satisfy relations which are locally similar to the $\mathfrak{hs}$ algebra of $\mathfrak{so}(4,1)$, while the X^a generate the underlying space.

3.6. Semi-classical limit and Poisson calculus

Now consider the semi-classical limit of fuzzy H^4_n, which is obtained for large n, and is indicated by $\sim$. Then $X^a \sim x^a$ and $\Theta^{ab} \sim \theta^{ab}$, and the above relations on $\mathcal{H}^4_n$ reduce to

$$x_a x^a = -R^2, \tag{3.58a}$$

$$\theta^{ab} x_b = 0\,, \tag{3.58b}$$

$$\epsilon_{abcde}\theta^{ab}x^c = nr\theta_{de} \sim 2R\theta_{de}\,, \tag{3.58c}$$

[9] We do not claim that for example the algebra of functions generated by $P^a = \mathcal{M}^{a4}$ is truncated at order n; this is not the case. The claim is that all Hilbert-Schmidt operators can be written in the above way.

$$\gamma^{bb'} := \eta_{aa'}\theta^{ab}\theta^{a'b'} = \frac{L_{NC}^4}{4} P^{bb'} , \tag{3.58d}$$

where the scale of non-commutativity is

$$L_{NC}^4 := \theta^{ab}\theta_{ab} = 4r^2R^2 . \tag{3.59}$$

Here

$$P^{ab} = \eta^{ab} + \frac{1}{R^2}x^a x^b \qquad \text{with} \quad P^{ab}x_b = 0 \quad \text{and} \quad P^{ab}P^{bc} = P^{ac} \tag{3.60}$$

is the Euclidean projector on H^4 (recall that H^4 is a Euclidean space). Hence the algebra of functions on fuzzy H_n^4 reduces for large n to the algebra of functions

$$\mathrm{End}(\mathcal{H}_n) \sim \mathcal{C}(\mathbb{C}P^{1,2}) = \oplus_s \mathcal{C}^s \tag{3.61}$$

on the classical Poisson manifold $\mathbb{C}P^{1,2}$, as described in section 2.1. We denote the eigenspaces of $\mathcal{S}^2$ again with $\mathcal{C}^s$, which are now modules over the algebra $\mathcal{C}^0 = \mathcal{C}(H^4)$ of functions on H^4, thus encoding the structure of a bundle over H^4. From now on we work in the semi-classical limit. The bundle structure can be made more explicit by writing

$$\theta^{ab} = \eta_i^{ab} J^i \tag{3.62}$$

as in (2.25), where η_i^{ab} are the tangential self-dual t'Hooft symbols; "tangential" follows from $x_a\theta^{ab} = 0$. The J^i transform as vectors of the local $SU(2)_L \subset SO(4)$, and describe the internal S^2 fiber.

Derivatives. It is useful to define the following derivations (cf. [2])

$$\eth^a\phi := -\frac{1}{r^2R^2}\theta^{ab}\{x_b,\phi\} = \frac{1}{r^2R^2}x_b\{\theta^{ab},\phi\}, \qquad \phi \in \mathcal{C} , \tag{3.63}$$

which are tangential $x^a\eth_a = 0$, satisfy the Leibniz rule, and are $SO(4,1)$-covariant. Equivalently,

$$\boxed{\{x^a,\cdot\} = \theta^{ab}\eth_b .} \tag{3.64}$$

In particular, the following holds:

$$\begin{aligned} \eth^a x^c &= -\frac{1}{r^2R^2}\theta^{ab}\{x_b,x^c\} = P_T^{ac} \\ [\eth_a,\eth_b]\phi &= -\frac{1}{r^2R^2}\{\theta_{ab},\phi\} \end{aligned} \tag{3.65}$$

as shown in appendix D. The first line shows that $\eth$ act as isometries on functions, such that the $\eth_a$ can be viewed as a set of five Killing vector fields on H^4 with Lie bracket given by (3.65). Furthermore,

$$\begin{aligned} \eth^a\theta^{cd} &= \frac{1}{r^2R^2}\theta^{ab}\{x_b,\theta^{cd}\} = -\frac{1}{R^2}\theta^{ab}(\eta^{bc}x^d - \eta^{bd}x^c) \\ &= \frac{1}{R^2}(-\theta^{ac}x^d + \theta^{ad}x^c). \end{aligned} \tag{3.66}$$

We also note that the $SO(4,1)$ rotations of scalar functions are generated by $\{\mathcal{M}^{ab},\cdot\}$, which can be written as

$$\{\mathcal{M}^{ab},\phi\} = -(x^a\eth_b - x^b\eth_a)\phi, \qquad \phi \in \mathcal{C}^0 . \tag{3.67}$$

To see this, it suffices to verify the action on the x^c generators,

$$\{\mathcal{M}^{ab}, x^c\} = -(x^a \eth_b - x^b \eth_a)x^c = -(x^a P^{bc} - x^b P^{ac}) = -(x^a \eta^{bc} - x^b \eta^{ac}) \tag{3.68}$$

since both sides are derivations. Finally, the semi-classical limit of the $\Box$ operator (3.6) can be expressed in terms of the derivatives as follows:

$$\begin{aligned}\Box\phi &= -\{x_a, \{x^a, \phi\}\} = -\{x_a, \theta^{ab}\eth_b\phi\} \\ &= -\{x_a, \theta^{ab}\}\eth_b\phi - \theta^{ab}\{x_a, \eth_b\phi\} \\ &= r^2\{\mathcal{M}^{ab}, x_a\}\eth_b\phi - \theta^{ab}\theta^{ac}\eth_b\partial_c\phi \\ &= -r^2R^2P^{ab}\eth_a\eth_b\phi\end{aligned} \tag{3.69}$$

for any $\phi \in \mathcal{C}$.

Connection. We define an $SO(4,1)$-covariant connection on the module $\mathcal{C}$ (4.1) by [2]

$$\nabla = P_T \circ \eth \tag{3.70}$$

so that for $\nabla_a \equiv \nabla_{\eth_a}$

$$\begin{aligned}\nabla_a\phi_b &= \partial_a\phi_b - \frac{1}{R^2}x_b\phi_a, \\ \nabla_a\phi_{bc} &= \partial_a\phi_{bc} - \frac{1}{R^2}(x_b\phi_{ac} + x_c\phi_{ba})\end{aligned} \tag{3.71}$$

etc. if ϕ_a, ϕ_{ab} are tangential. Comparing with (3.66) and using (3.67) it follows that the connection is compatible with θ^{ab}, i.e.

$$\nabla\theta^{ab} = 0, \qquad \nabla\{f, g\} = \{\nabla f, g\} + \{f, \nabla g\} \tag{3.72}$$

and $\nabla_a P_{bc} = 0$. The associated curvature

$$\mathcal{R}_{ab} := \mathcal{R}[\eth_a, \eth_b] = [\nabla_a, \nabla_b] - \nabla_{[\eth_a, \eth_b]} \tag{3.73}$$

is computed in appendix D.15, and reduces to the Levi–Civita connection on tensor fields. Thus H^4_n is a quantum space which is fully $SO(4,1)$-covariant, and we have found a calculus which is defined solely in terms of the Poisson bracket, i.e. the semi-classical limit of matrix commutators. This is very important for the present non-commutative framework.

Averaging over the fiber. There exists a canonical map

$$\begin{aligned}&[\cdot]_0 : \mathcal{C}(\mathbb{C}P^{1,2}) \to \mathcal{C}(H^4) \\ &\qquad f(\xi) \mapsto f(x) = \int\limits_{S^2} f(\xi)\end{aligned} \tag{3.74}$$

defined by integrating over the fiber at each $x \in H^4$. This projects the functions on the total space to functions on the base space. On fuzzy S^4_N, this averaging can be defined in terms of a $SO(5)$-invariant projection to some sub-space of $\mathrm{End}(\mathcal{H})$. For H^4_n, $[\cdot]_0$ is nothing but the projection to $\mathcal{S}^2 = 0$ i.e. to $\mathcal{C}^0$, as discussed below.

Explicitly, the averaging $[\cdot]_0$ over the internal S^2 is given by

$$\left[\theta^{ab}\theta^{cd}\right]_0 = \frac{1}{12}L_{NC}^4(P^{ac}P^{bd} - P^{bc}P^{ad} + \varepsilon^{abcde}\frac{1}{R}x^e)$$
$$= \frac{r^2R^2}{3}(P^{ac}P^{bd} - P^{bc}P^{ad} + \varepsilon^{abcde}\frac{1}{R}x^e)\,. \tag{3.75}$$

One can generalize the averaging to higher powers of θ^{ab}, e.g. [2]

$$[\theta^{ab}\theta^{cd}\theta^{ef}\theta^{gh}]_0 = \frac{3}{5}\Big([\theta^{ab}\theta^{cd}]_0[\theta^{ef}\theta^{gh}]_0 + [\theta^{ab}\theta^{ef}]_0[\theta^{cd}\theta^{gh}]_0 + [\theta^{ab}\theta^{gh}]_0[\theta^{cd}\theta^{ef}]_0\Big)\,. \tag{3.76}$$

Alternatively, one could proceed to define a star product for functions on H^4, which is presumably commutative, but not associative, in analogy to the case of S_N^4 [11]. On the other hand, for $n = 0$ there is nothing to project, and the full algebra of functions on H^4 is non-commutative and associative without extra generators.

Integration. As for any quantized coadjoint orbit, the trace on $\mathrm{End}(\mathcal{H})$ corresponds to the integral over the underlying symplectic space, defined by the symplectic volume form. Explicitly,

$$\mathrm{Tr}\mathcal{Q}(\phi) = \int d\mu\,\phi = \int\limits_{H^4} \rho[\phi]_0\,, \qquad \rho \mathrel{\hat{=}} \frac{\dim(\mathcal{H})}{\mathrm{Vol}(H^4)} \tag{3.77}$$

replacing the ill-defined fraction $\frac{\dim(\mathcal{H})}{\mathrm{Vol}(H^4)}$ with the symplectic volume form $d\mu$, which reduces to ρ on H^4. This is best seen via coherent states (3.32). We will often drop $d\mu$ and $\mathcal{Q}$ in the semi-classical limit. Finally, note that the $\eth^a$ are *not* self-adjoint under the integral, but

$$\int \eth_a f\, g = -\int f\,\eth_a g + \frac{1}{\theta R^2}\int f\{x_b, \theta^{ab}\}g = -\int f\,\eth_a g - \frac{4}{R^2}\int x^a f g \tag{3.78}$$

using $\{x_b, \theta^{ba}\} = 4r^2x^a$.

4. Functions, tensors and higher-spin modes

We have seen that the algebra $\mathrm{End}(\mathcal{H}_n)$ of fuzzy H_n^4 reduces in the semi-classical limit to the algebra of functions on $\mathbb{C}P^{1,2}$. The results of section 3.5 provide a more detailed decomposition of $\mathcal{C}$ into modules (3.55)

$$\boxed{\mathcal{C} = \bigoplus_{s=0}^{\infty}\mathcal{C}^s \quad \ni \phi^s_{a_1\ldots a_s;b_1\ldots b_s}(x)\, m^{a_1b_1}\ldots m^{a_sb_s} \equiv \phi^s_{\underline{\beta}}(x)\,\Xi^{\underline{\beta}},} \tag{4.1}$$

over the algebra of functions $\mathcal{C}^0$ on H^4, due to (3.46). This means that $\mathcal{C}$ is a bundle over H^4, whose structure is determined by the constraints (2.17), (2.21) and (2.24). An explicit description is given by the one-to-one map[10]

[10] Note that $\Gamma^{(s)}H^4$ is not a module over $\mathcal{C}^0$, hence this is not a module isomorphism. In [2], a different convention was used for the map $\phi_{a_1\ldots a_s}(x) \leftrightarrow \phi^{(s)}$. The present convention avoids the appearance of square-roots of Casimirs in this map.

$$\boxed{\begin{aligned} \Gamma^{(s)}H^4 &\to \mathcal{C}^s \\ \phi^{(s)}_{a_1\ldots a_s}(x) &\mapsto \phi^{(s)} = \{x^{a_1}, \ldots \{x^{a_s}, \phi^{(s)}_{a_1\ldots a_s}\}\ldots\}\,. \end{aligned}} \tag{4.2}$$

Here $\Gamma^{(s)}H^4$ denotes the space of totally symmetric, traceless, divergence-free rank s tensor fields on H^4, which are identified with (symmetric tangential divergence-free traceless) tensor fields $\phi^{(s)}_{a_1\ldots a_s}$ with $SO(4,1)$ indices, as discussed in section 4.2 and in [2]. The inverse map of (4.2) (up to normalization) can be given by

$$\mathcal{C}^s \ni \phi^{(s)} \mapsto \{x^{a_1}, \ldots \{x^{a_s}, \phi^{(s)}\}\ldots\}_0 \ \in \Gamma^{(s)}H^4 \tag{4.3}$$

which is symmetric due to $[\cdot]_0$, as well as traceless, divergence-free and tangential. These statements are analogous to the results in [2].

Some comments on the map (4.2) are in order. We show in sections 4.0.1–4.0.3 that pure divergence modes would be mapped to zero by (4.2). Injectivity will be shown below by establishing (4.3). To see surjectivity, it suffices to consider the vicinity of a chosen reference point, for instance (2.6). Then polynomial functions suffice to approximate any element in $\mathcal{C}^s$. Then the $\mathfrak{so}(4,2)$ representation theory allows to characterize all polynomials in $\mathcal{C}^s$ uniquely by Young diagrams, as explained in detail in [2, section 3]. These in turn are captured by the map (4.2), and an alternative inverse map can be used [2]

$$\mathcal{C}^s \ni \phi^{(s)}_{a_1\ldots a_s;b_1\ldots b_s}(x)\, m^{a_1b_1}\ldots m^{a_sb_s} \mapsto \phi^{(s)}_{a_1\ldots a_s;b_1\ldots b_s}(x)\, x^{b_1}\ldots x^{b_s} \ \in \Gamma^{(s)}H^4\,, \tag{4.4}$$

which is equivalent to (4.3) up to normalization.

Hence $\mathcal{C}^s$ encodes one and only one irreducible spin s field on H^4, given by square-integrable tensor fields on H^4. The generators $\Xi^{\underline{\beta}}$ form a basis of irreducible totally symmetric polynomials in m^{ab}, i.e. of Young tableaux

$$\oplus \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline \end{array} \cong \mathfrak{hs} := \oplus_{s=1}^{\infty} \Xi^{s,\underline{\beta}}\,. \tag{4.5}$$

As in [2], $\mathfrak{hs}$ is closely related to the higher-spin algebra of Vasiliev theory.[11] Hence $\mathcal{C}$ can be viewed as functions on H^4 taking values in $\mathfrak{hs}$.

4.0.1. Spin 1 modes

The unique spin 1 field is encoded in $\phi_{ab}m^{ab}$. According to the above statements, it can be expressed in terms of a tangential, divergence-free tensor field $\phi_a \in \mathcal{C}^0$ on H^4, i.e.

$$x^a\phi_a = 0 = \eth^a\phi_a\,. \tag{4.6}$$

Given such a ϕ_a, we define

$$\begin{aligned} \phi^{(1)} &:= \{x^a, \phi_a\} = \theta^{ab}\eth_b\phi_a = \frac{1}{2}\theta^{ab}\mathcal{F}_{ab} \quad \in \mathcal{C}^1, \\ \mathcal{F}_{ab} &= \eth_b\phi_a - \eth_a\phi_b \end{aligned} \tag{4.7}$$

which encodes the field strength of the vector field. This is not tangential, but

$$x^a\mathcal{F}_{ab} = x^a\eth_b\phi_a - x^a\eth_a\phi_b = x^a\eth_b\phi_a = -\phi_b \tag{4.8}$$

[11] Note that $\mathcal{H}_0$ is a minirep of $SO(4,2)$ but not of $SO(4,1)$. This explains why we get an extension of Vasiliev's $\mathfrak{hs}$ algebra by functions of X.

using (D.1). Conversely, the “potential” $\phi_a(x)$ is recovered from $\phi^{(1)}$ via a projection

$$\boxed{-\{x_a, \phi^{(1)}\}_0 = \alpha_1(\Box - 2r^2)\phi_a, \qquad \alpha_1 = \frac{1}{3}} \tag{4.9}$$

where $\phi^{(1)} = \{x^c, \phi_c\}$ for a tangential, divergence-free $\phi_a \in \mathcal{C}^0$. The derivation of (4.9) is detailed in (D.18)–(D.20). The generalization of this formula for higher-spin is discussed below. If ϕ_a is an irrep of $SO(4,1)$, we may abbreviate this as

$$-\{x_a, \phi^{(1)}\}_0 =: \hat{\alpha}_1\phi_a \tag{4.10}$$

where $\hat{\alpha}_1$ is the value of $\alpha_1(\Box - 2r^2)$ on ϕ_a.

Pure gauge modes. Finally, one can verify that for $\tilde{\phi}_a = \eth_a\phi$, the associated “field strength” tensor is $\mathcal{F}_{ab} \propto \{\theta^{ab}, \phi\}$, but the field strength form $\phi^{(1)}$ vanishes identically:

$$\phi^{(1)} = \{x^a, \eth_a\phi\} = 0 \tag{4.11}$$

using (D.3). This expresses the gauge invariance (or irreducibility) of $\phi^{(1)}$.

4.0.2. Spin 2 modes

Similarly, spin 2 modes can be realized in terms of a tangential, divergence-free, traceless, symmetric rank 2 tensor $\phi_{ab}(x) = \phi_{ba}(x) \in \mathcal{C}^0$, i.e.

$$x^a\phi_{ab} = 0 = \eth^a\phi_{ab} = \eta^{ab}\phi_{ab}\ . \tag{4.12}$$

We define the associated “potential form”

$$\phi^{(2)}_a = \{x^b, \phi_{ab}\} = \theta^{bc}\eth_c\phi_{ab} = -\omega_{a;cb}\theta^{cb} \qquad \in \mathcal{C}^1\ , \tag{4.13}$$

which can be viewed as $\mathfrak{so}(4,1)$-valued one-form with

$$\omega_{a;cb} = \frac{1}{2}(\eth_c\phi_{ab} - \eth_b\phi_{ac})\ . \tag{4.14}$$

Note that $\phi^{(2)}_c$ is indeed tangential,

$$x^c\phi^{(2)}_c = x^c\{x^a, \phi_{ca}\} = -\{x^a, x^c\}\phi_{ca} = 0\ . \tag{4.15}$$

The $\mathfrak{so}(4)$-valued components of $\phi^{(2)}_a$ correspond to the spin connection, while its translational components

$$x^c\omega_{a;cb} = -\frac{1}{2}x^c\eth_b\phi_{ac} = \frac{1}{2}\phi_{ab} \tag{4.16}$$

reduce to ϕ_{ab}, as on fuzzy S^4_N [2]. The “field strength form” corresponding to $\phi^{(2)}_a$ is

$$\begin{aligned}
\phi^{(2)} = \{x^a, \phi^{(2)}_a\} &=: \frac{1}{2}\theta^{ad}\mathcal{R}_{ad}[\phi] \\
&= -\theta^{ad}\eth_d(\omega_{a;cb}\theta^{cb}) \ = -\theta^{cb}\theta^{ad}\eth_d\omega_{a;cb} - \theta^{ad}\omega_{a;cb}\eth_d\theta^{cb} \\
&= \frac{1}{2}\theta^{ad}\theta^{cb}(\eth_a\omega_{d;cb} - \eth_d\omega_{a;cb}) \\
&=: \frac{1}{2}\theta^{ad}\theta^{bc}\mathcal{R}_{ad;bc}[\phi] \qquad \in \mathcal{C}^2
\end{aligned} \tag{4.17}$$

noting that the $\eth_d\theta^{cb}$ terms drop out for traceless, tangential ϕ_{ab}, using (3.66). This encodes the linearized Riemann curvature tensor associated to ϕ_{ab},

$$\begin{aligned}\mathcal{R}_{ad}[\phi] &:= -\eth_a\phi^{(2)}_d + \eth_d\phi^{(2)}_a = \mathcal{R}_{ad;bc}\theta^{bc} \quad \in \mathcal{C}^1,\\ \mathcal{R}_{ad;bc} &= \eth_a\omega_{d;bc} - \eth_d\omega_{a;bc}\\ &= \frac{1}{2}(\eth_d\eth_c\phi_{ab} - \eth_a\eth_c\phi_{db} - \eth_d\eth_b\phi_{ac} + \eth_a\eth_b\phi_{dc})\,. \end{aligned} \tag{4.18}$$

Although the $\eth_e$ do not commute among another, their commutator is radial due to (3.67), i.e.

$$P\mathcal{R}_{ad;bc} - P\mathcal{R}_{bc;ad} = 0\,. \tag{4.19}$$

Hence the tangential components of $P\mathcal{R}_{ae;bc}[\phi]$ coincide with the usual linearized Riemann tensor. The connection form $\phi^{(2)}_c$ (i.e. ω) is recovered by a projection

$$\boxed{-\{x_a, \phi^{(2)}\}_1 = \alpha_2(\Box - 2r^2)\phi^{(2)}_a \quad \in \mathcal{C}^1, \qquad \alpha_2 = \frac{2}{5}} \tag{4.20}$$

generalizing (4.9). Here, we defined $\phi^{(2)} = \{x^b, \{x^c, \phi_{bc}\}\}$ for a tangential, divergence-free, traceless $\phi_{ab} \in \mathcal{C}^0$. Similarly to the spin 1 case, (4.20) could be obtained via formula (3.76); however, we provide a more transparent derivation by means of an inner product below. If the underlying tensor ϕ_{ab} is an irrep of $SO(4,1)$, we may abbreviate this as

$$-\{x_a, \phi^{(2)}\}_0 =: \hat{\alpha}_2\phi^{(2)}_a \tag{4.21}$$

where $\hat{\alpha}_2$ is the value of $\alpha_2(\Box - 2r^2)$ on $\phi^{(2)}_a$.

Spin 2 pure gauge modes. Again, consider a pure gauge rank 2 tensor

$$\tilde{\phi}^{(1)}_{ab} = \nabla_a\phi_b + \nabla_b\phi_a \tag{4.22}$$

which is tangential and traceless (provided $\eth^a\phi_a = 0$), but no longer divergence-free. Then

$$\begin{aligned}\tilde{\phi}^{(1)}_a &:= \{x^b, \tilde{\phi}^{(1)}_{ab}\} = \{x^b, \eth_a\phi_b + \eth_b\phi_a - \frac{1}{R^2}(x^a\phi_b + x^b\phi_a)\}\\ &= \{x^b, \eth_a\phi_b\} - \frac{1}{R^2}\{x^b, x^a\phi_b\}\\ &= \eth_a\phi^{(1)} + \frac{2}{R^2}\theta^{ac}\phi_c \end{aligned} \tag{4.23}$$

using (D.7) and $\phi^{(1)} = \{x^a, \phi_a\}$. This satisfies

$$\{x^a, \tilde{\phi}^{(1)}_a\} = \{x^a, \eth_a\phi^{(1)} + \frac{2}{R^2}\theta^{ac}\phi_c\} = \frac{2}{R^2}\{x^a, \theta^{ac}\phi_c\} = 0 \tag{4.24}$$

using (5.30), which expresses the gauge invariance of $\phi^{(2)}$.

4.0.3. Spin s modes and Young diagrams

As observed above, elements in $\mathcal{C}^s$ can be identified with totally symmetric, traceless, divergence-free rank s tensor fields $\phi_{a_1\ldots a_s}$ on H^4 via

$$\phi^{(s)} = \{x^{a_1}, \ldots \{x^{a_s}, \phi_{a_1\ldots a_s}\}\ldots\} \in \mathcal{C}^s\,. \tag{4.25}$$

It is useful to define also the mixed spin s objects, such as the “connection $(2s-1)$-form”

$$\phi_a^{(s)} = \{x^{a_1}, \ldots, \{x^{a_{s-1}}, \phi_{a_1 \ldots a_{s-1} a}\} \ldots\} \ \in \mathcal{C}^{s-1} \tag{4.26}$$

which are all tangential and associated to the underlying irreducible rank s tensor field. Then the "field strength" form can be written as

$$\begin{aligned}\phi^{(s)} &= \{x^a, \phi_a^{(s)}\} =: \frac{1}{2}\mathcal{R}_{ad}[\phi]\theta^{ad} \\ &= \theta^{a_1 b_1}\eth_{b_1} \ldots \theta^{a_s b_s}\eth_{b_s}\phi_{a_1 \ldots a_s} = \theta^{a_1 b_1} \ldots \theta^{a_s b_s}\eth_{b_1} \ldots \eth_{b_s}\phi_{a_1 \ldots a_s} \\ &=: \mathcal{R}_{a_1 \ldots a_s; b_1 \ldots b_s}(x)\, \theta^{a_1 b_1} \ldots \theta^{a_s b_s} \quad \equiv \ \mathcal{R}_{\underline{\alpha}}(x)\, \Xi^{\underline{\alpha}} \qquad \in \mathcal{C}^s\end{aligned} \tag{4.27}$$

noting that the $\eth\theta^{\cdots}$ terms drop out for traceless tangential $\phi_{a_1 \ldots a_s}$, using (3.66). Here

$$\begin{aligned}\mathcal{R}_{ad}[\phi] &:= -\eth_a \phi_d^{(s)} + \eth_d \phi_a^{(s)} \\ \mathcal{R}_{b_1 \ldots b_s; a_1 \ldots a_s}(x) &= \mathcal{P}\eth_{a_1} \ldots \eth_{a_s}\phi_{b_1 \ldots b_s}\end{aligned} \tag{4.28}$$

is some antisymmetrized derivatives corresponding to some two row rectangular Young projector $\mathcal{P} \sim \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline\end{array}$, which can be regarded as linearized higher-spin curvature. We will show below that the potential $\phi_a^{(s)}$ is then recovered from the following projection

$$\boxed{-\{x_a, \phi^{(s)}\}_{s-1} = \alpha_s(\Box - 2r^2)\phi_a^{(s)} \qquad \in \mathcal{C}^{s-1}.} \tag{4.29}$$

If the underlying $\phi_{a_1 \ldots a_s} \in \mathcal{C}^0$ is an irrep of $SO(4,1)$, we may abbreviate this as

$$-\{x_a, \phi^{(s)}\}_{s-1} =: \hat{\alpha}_s \phi_a^{(s)} \tag{4.30}$$

where $\hat{\alpha}_s$ is the value of $\alpha_s(\Box - 2r^2)$ on $\phi_a^{(s)}$.

Pure gauge modes. Finally, one can verify that a pure gauge rank s tensor

$$\tilde{\phi}^{(s-1)}_{a_1 \ldots a_s} = \nabla_{(a_s}\phi_{a_1 \ldots a_{s-1})} \tag{4.31}$$

drops out from the field strength form,

$$\{x^{a_1}, \ldots, \{x^{a_s}, \tilde{\phi}^{(s-1)}_{a_1 \ldots a_s}\} \ldots\} = 0\,. \tag{4.32}$$

As before, this is a manifestation of the gauge invariance of $\phi^{(s)}$. One way to see this is to move ∇ out of the brackets using (3.72) and, finally, use (D.4).

One may wonder about the meaning of the infinitesimal transformations

$$\phi^{(s)} \mapsto \{\Lambda, \phi^{(s)}\}\,. \tag{4.33}$$

These correspond to symplectomorphisms on $\mathbb{C}P^{1,2}$ generated by the Hamiltonian vector field $\{\Lambda, \cdot\}$, which mix the different spin modes in a non-trivial way. They do not correspond to the above pure gauge modes (4.31), but see section 5.9.

4.1. Inner product and quadratic action

It is interesting and useful to compute the inner product (3.43) of the above spin s fields $\phi^{(s)}$ defined by the trace in End($\mathcal{H}$). In the spin 1 case, consider the quadratic form

$$\begin{aligned}\int \phi^{(1)}\phi^{(1)} &= \frac{1}{4}\int [\theta^{ab}\theta^{cd}]_0 \mathcal{F}_{ab}\mathcal{F}_{cd} \\ &= \frac{r^2 R^2}{12}\int (2P^{ac}P^{bd} + \frac{x^f}{R}\varepsilon^{abcdf})\mathcal{F}_{ab}\mathcal{F}_{cd}\end{aligned} \tag{4.34}$$

which looks like the action for self-dual (abelian) Yang–Mills. In the spin 2 case, consider the analogous quadratic form

$$\begin{aligned}
&\int \phi^{(2)}\phi^{(2)} \\
&= \frac{1}{4}\int [\theta^{ae}\theta^{bc}\theta^{a'e'}\theta^{b'c'}]_0 \mathcal{R}_{ae;bc}[\phi]\mathcal{R}_{a'e';b'c'}[\phi] \\
&= \frac{3}{10}\int [\theta^{ae}\theta^{a'e'}]_0[\theta^{bc}\theta^{b'c'}]_0 \mathcal{R}_{ae;bc}[\phi]\mathcal{R}_{a'e';b'c'}[\phi] \\
&= \frac{1}{30}\int (2P^{aa'}P^{ee'} + \frac{x^f}{R}\varepsilon^{aea'e'f})(2P^{bb'}P^{cc'} + \frac{x^f}{R}\varepsilon^{bcb'c'f})\mathcal{R}_{ae;bc}[\phi]\mathcal{R}_{a'e';b'c'}[\phi] \\
&= \frac{2}{15}\int P^{aa'}P^{ee'}P^{bb'}P^{cc'}\mathcal{R}_{ae;bc}[\phi]\mathcal{R}_{a'e';b'c'}[\phi] \quad + \text{topological terms}
\end{aligned} \tag{4.35}$$

because $[\phi^{(2)}]_0 = 0$. Note that we used the symmetries (4.19) of $\mathcal{R}_{ae;bc}$ or rather of its tangential part $P\mathcal{R}_{ae;bc}$, as the radial contributions drop out anyway. We observe that (4.35) is a (self-dual) linearized quadratic gravity action,[12] which can be written in terms of the $\mathcal{R}_{ab}$ "forms" as follows:

$$\begin{aligned}
\int \phi^{(2)}\phi^{(2)} &= \frac{1}{4}\int [\theta^{ae}\theta^{bc}\theta^{a'e'}\theta^{b'c'}]_0 \mathcal{R}_{ae;bc}\mathcal{R}_{a'e';b'c'} \\
&= \frac{3}{10}\int [\theta^{ae}\theta^{a'e'}]_0 \mathcal{R}_{ae;bc}\theta^{bc}\mathcal{R}_{a'e';b'c'}\theta^{b'c'} \\
&= \frac{3}{10}\int [\theta^{ae}\theta^{a'e'}]_0 \mathcal{R}_{ae}\mathcal{R}_{a'e'} \\
&= \frac{2r^2R^2}{5}\int (P^{ac}P^{bd} - P^{ad}P^{bc} + \frac{x^e}{R}\varepsilon^{abcde})\eth_b\phi_a^{(2)}\eth_d\phi_c^{(2)} \,.
\end{aligned} \tag{4.36}$$

Similarly for spin s, we have

$$\begin{aligned}
\int \phi^{(s)}\phi^{(s)} &= \int [\theta^{ae}\theta^{a'e'}\mathcal{R}_{ae}\mathcal{R}_{a'e'}]_0 \\
&= \alpha_s r^2R^2 \int (P^{ac}P^{bd} - P^{ad}P^{bc} + \frac{x^e}{R}\varepsilon^{abcde})\eth_b\phi_a^{(s)}\eth_d\phi_c^{(s)}
\end{aligned} \tag{4.37}$$

which is again some self-dual quadratic Fronsdal-type higher-spin action [29]. The factor α_s will be determined below. This suggests that a matrix model based on a single $\phi \in \mathrm{End}(\mathcal{H})$ should define some higher-spin theory, which is however expected to be more or less trivial. Nevertheless it would be interesting to study the action defined by higher-order polynomials, and to understand its relation with Vasiliev's theory [30]. In the remainder of this paper, we will show how a non-trivial higher-spin gauge theory arises from multi-matrix models.

4.1.1. Projections, positivity and determination of α_s

Now consider the spin s modes $\phi^{(s)} \in \mathcal{C}^s$ as above, determined by some irreducible rank s tensor field on H^4. We have seen that this in one-to-one correspondence to a spin s potential $\phi_a^{(s)} \in \mathcal{C}^{s-1}$ as above. Then

[12] The topological terms are the linearized Pontryagin and Euler class (i.e. Gauss–Bonnet term).

$$-\int \phi_a^{(s)}\{x_a, \phi^{(s)}\}_{s-1} = -\int \phi_a^{(s)}\{x_a, \phi^{(s)}\} = \int \{x^a, \phi_a^{(s)}\}\phi^{(s)} = \int \phi^{(s)}\phi^{(s)} \,. \tag{4.38}$$

This provides the following relations:

Spin 1 case. For spin $s = 1$, the projection $\{x_a, \phi^{(s)}\}_0$ in (4.38) was computed in (4.9), which gives

$$\int \phi^{(1)}\phi^{(1)} = \alpha_1 \int \phi_a^{(1)}(\Box - 2r^2)\phi_a^{(1)} \ \geq 0, \tag{4.39}$$

for Hermitian $\phi^{(1)}$ Therefore

$$\alpha_1 = \frac{1}{3}\,, \tag{4.40}$$

and in particular $\Box - 2r^2$ is *positive* on $\mathcal{C}^0$.

Spin 2 case. We can evaluate the right-hand side of (4.38) using (4.36) as

$$\begin{aligned}
\int \phi^{(2)}\phi^{(2)} &= \frac{2r^2R^2}{5}\int P^{ac}\eth_b\phi_a^{(2)}\eth^b\phi_c^{(2)} - \eth^d\phi_a^{(2)}\eth^a\phi_d^{(2)} + \frac{x^e}{R}\varepsilon^{abcde}\eth_b\phi_a^{(2)}\eth_d\phi_c^{(2)}\\
&= \frac{2r^2R^2}{5}\int -\phi_a^{(2)}\eth_b\eth^b\phi_a^{(2)} + \frac{1}{R^2}\phi_b^{(2)}\phi_b^{(2)} - \frac{4}{R^2}\phi_a^{(2)}\phi_a^{(2)}\\
&\qquad + \frac{1}{r^2R^2}\phi_a^{(2)}\Big(\{\theta^{ad}, \phi_d^{(2)}\} - \frac{1}{2R}\varepsilon^{abdce}x^e\{\theta_{bd}, \phi_c^{(2)}\}\Big)\\
&= \alpha_2\int \phi_a^{(2)}(\Box - 2r^2)\phi_a^{(2)}
\end{aligned} \tag{4.41}$$

using (3.78), (D.22), the self-duality relation (D.23) and (D.24). Therefore

$$\alpha_2 = \frac{2}{5}\,. \tag{4.42}$$

This holds in fact for any tangential divergence-free $\phi_c \in \mathcal{C}^1$. Together with (4.38), this establishes the formula (4.20). On the other hand, (4.40) and (4.39) implies also e.g.[13]

$$\int \phi_a^{(2)}\phi_a^{(2)} = \frac{1}{3}\int \phi_{ab}(\Box - 2r^2)\phi_{ab} \tag{4.43}$$

if $\phi_{ab} \in \mathcal{C}^0$ is divergence-free, traceless and tangential (by fixing one index).

Generic spin s case. In the generic case, we obtain similarly

$$\int \phi^{(s)}\phi^{(s)} = \int \phi_a^{(s)}\hat{\alpha}_s\phi_a^{(s)}, \qquad \hat{\alpha}_s\phi^{(s)} = \alpha_s(\Box - 2r^2)\phi_a^{(s)}\,, \tag{4.44}$$

$$\int \phi^{(s)}\phi^{(s)} = \int \phi_{a_1\ldots a_s}^{(s)}\hat{\alpha}_s\ldots\hat{\alpha}_2\hat{\alpha}_1\phi_{a_1\ldots a_s}^{(s)}\,. \tag{4.45}$$

Explicit expressions for α_s for $s \geq 3$ could be computed similarly but are not required for our purposes.

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[13] Since $\phi_a^{(2)} = \{x^c, \phi_{cb}\}$ for tangential traceless divergence-free $\phi_{ab} \in \mathcal{C}^0$; the index a is irrelevant here.

4.2. Local decomposition

Finally consider any point on H^4, for instance the reference point (2.6). We denote the four tangential coordinates with x^μ, and the time-like coordinate on $\mathbb{R}^{4,1}$ with x^0. Then the $\mathfrak{so}(4,1)$ generators decompose (locally) into $\mathfrak{so}(4)$ generators $m^{\mu\nu}$, and the remaining translation generator by $p^\mu = m^{\mu 0}$. We can then decompose e.g. the spin $s=1$ modes locally as

$$\phi_{ab}(x)m^{ab} = \phi_\mu(x)\, p^\mu + \phi_{\mu\nu}(x)m^{\mu\nu} \qquad \in \mathcal{C}^1 \tag{4.46}$$

and similar for higher-spin. From this point of view, the main lesson of the above results is that the $\phi_\mu(x)$ and $\phi_{\mu\nu}(x)$ are *not* independent fields, but determined by the same irreducible spin 1 field $\phi_a(x)$, and similarly for higher-spin fields. For generalized fuzzy spaces these constraints may disappear, as considered in [3]. For the basic spaces H^4_n and for S^4_N [2], the formalism developed above takes these constraints properly into account.

5. Matrix model realization and fluctuations

Now consider the IKKT matrix models with mass term,

$$S[Y] = \frac{1}{g^2}\mathrm{Tr}\Big([Y^a, Y^b][Y^{a'}, Y^{b'}]\eta_{aa'}\eta_{bb'} - \mu^2 Y_a Y^a\Big)\,. \tag{5.1}$$

Here $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$ is interpreted as Minkowski metric of the target space $\mathbb{R}^{1,D-1}$. The positive mass $\mu^2 > 0$ should ensure stability. The above model leads to the classical equations of motion

$$\Box_Y Y^a + \frac{1}{2}\mu^2 Y^a = 0 \tag{5.2}$$

where

$$\Box_Y = \big[Y^a, [Y_a, \cdot]\big] \sim -\{y^a, \{y_a, \cdot\}\} \tag{5.3}$$

plays the role of the Laplacian. Note that (5.2) are precisely the equation of motions for the IKKT model put forward in [31] after taking an IR cutoff into account.

5.1. Fuzzy H^4_n solution and tangential fluctuation modes

Consider the solution $Y^a = X^a$ of (3.6) corresponding to fuzzy H^4_n, and add fluctuations

$$Y^a = X^a + \mathcal{A}^a \tag{5.4}$$

on H^4_n. They naturally separate into tangential modes $x_a\mathcal{A}^a{=}0$ and radial modes $x_a\mathcal{A}^a{\neq}0$. The $SO(4,1)$-invariant inner product

$$\Big\langle \mathcal{A}^{(i)}, \mathcal{A}^{(j)}\Big\rangle := \int \mathcal{A}^{(i)}_a \mathcal{A}^{(j)}_b \eta^{ab} \tag{5.5}$$

is positive definite for (Hermitian) tangential $\mathcal{A}_a$ on H^4, and negative for the radial modes. Since $\mathcal{A}^a \in \mathrm{End}(\mathcal{H}_n)\otimes\mathbb{R}^5$, we expect four tangential fluctuation modes and one radial mode for each spin (except for spin 0), as for S^4_N [2]. Our strategy will be to remove the radial modes, and to find a useful basis of tangential modes in the semi-classical limit.

Intertwiners. Define the $SO(4,1)$ intertwiners

$$\begin{aligned}\mathcal{I}(\mathcal{A}^a) &:= \{\theta^{ab}, \mathcal{A}_b\}\\ \tilde{\mathcal{I}}(\mathcal{A}_a) &:= P_{aa'}\{\theta^{a'b}, \mathcal{A}_b\}\\ \mathcal{G}(\mathcal{A}^a) &:= \{x^a, \{x^b, \mathcal{A}_b\}\}\ .\end{aligned} \tag{5.6}$$

They are Hermitian w.r.t. the inner product (5.5), and tangential except for $\mathcal{I}$, noting that

$$\begin{aligned}x^a\,\mathcal{I}(\mathcal{A}_a) &= x^a\,\{\theta^{ac}, \mathcal{A}_c\} = -r^2R^2\,\eth^a\mathcal{A}_a\\ \eth^a\mathcal{I}(\mathcal{A}_a) &= \eth_a\{\theta^{ad}, \mathcal{A}_d\} = \frac{1}{r^2R^2}x_b\{\theta^{ab}, \{\theta^{ad}, \mathcal{A}_d\}\}\\ &= -\frac{1}{r^2R^2}\,x^b\,\mathcal{I}^2(\mathcal{A}_b)\ .\end{aligned} \tag{5.7}$$

The $SO(4,1)$ Casimir for the fluctuation modes can be expressed using $\mathcal{I}$ as follows:

$$\begin{aligned}C^2[\mathfrak{so}(4,1)]^{(\mathrm{full})}\mathcal{A}^a &= \frac{1}{2}([\mathcal{M}_{cd},\cdot] + M^{(5)}_{cd})^2\mathcal{A}^a\\ &= C^2[\mathfrak{so}(4,1)]^{(\mathrm{ad})}\mathcal{A}^a - 2r^{-2}\mathcal{I}(\mathcal{A}^a) + 4\\ &= (-r^{-2}\Box - 2r^{-2}\mathcal{I} + \mathcal{S}^2 + 4)\mathcal{A}^a\\ &= (R^2\eth\cdot\eth - 2r^{-2}\mathcal{I} + \mathcal{S}^2 + 4)\mathcal{A}^a\end{aligned} \tag{5.8}$$

using (3.44), and $C^2[\mathfrak{so}(4,1)] = 4$ for the vector representation $\mathbb{C}^5$. This can be seen by expressing $\mathcal{I}$ as follows:

$$-\theta(M^{(\mathrm{ad})}_{cd}\otimes M^{(5)}_{cd}\mathcal{A})^a \sim -\big(M^{(5)}_{cd}\big)^a_b\mathrm{i}\{\theta^{cd},\cdot\}\mathcal{A}^b = 2\{\theta_{ab}, \mathcal{A}^b\} = 2\mathcal{I}(\mathcal{A})^a\ . \tag{5.9}$$

Here

$$(M^{(5)}_{ab})^c_d = \mathrm{i}(\delta^c_b\eta_{ad} - \delta^c_a\eta_{bd}) \tag{5.10}$$

is the vector generator of $\mathfrak{so}(4,1)$, and $M^{(\mathrm{ad})}_{bc} = \mathrm{i}\{\mathcal{M}_{bc},\cdot\}$ denotes the representation of $\mathfrak{so}(4,1)$ induced by the Poisson structure on $\mathcal{S}^4$. As a check, we note that $C^{(\mathrm{full})}(x^a) = 0$, since $\mathcal{I}(x^a) = 4x^a$. This reflects the full $SO(4,1)$-invariance of the background x^a.

5.1.1. *Spin 0 modes*

Let $\phi \in \mathcal{C}^0$ be a spin 0 scalar field. There are two tangential spin 0 modes, which read

$$\begin{aligned}\mathcal{A}^{(1)}_a &= \eth_a\phi\ \in \mathcal{C}^0, \qquad \phi \in \mathcal{C}^0\ ,\\ \mathcal{A}^{(2)}_a &= \theta^{ab}\eth_b\phi\ = \{x^a, \phi\}\ \in \mathcal{C}^1\ .\end{aligned} \tag{5.11}$$

These modes satisfy

$$\begin{aligned}\{x^a, \mathcal{A}^{(1)}_a\} &= \{x^a, \eth_a\phi\} = 0\ ,\\ \{x^a, \mathcal{A}^{(2)}_a\} &= -\Box\phi\ ,\end{aligned} \tag{5.12}$$

using (D.3). Clearly only $\mathcal{A}^{(1)}_a$ is physical, while $\mathcal{A}^{(2)}_a$ is a pure gauge field. Let us compute the action of the $\mathcal{I}$ intertwiner; to start with

$$\begin{aligned}\mathcal{I}(\mathcal{A}_a^{(2)}) &:= \{\theta^{ab}, \{x^b, \phi\}\} = \{\{\theta^{ab}, x^b\}, \phi\}\} + \{x^b, \{\theta^{ab}, \phi\}\} \\ &= 4r^2\{x^a, \phi\} + r^2\{x^b, (x^a \eth^b - x^b \eth^a)\phi\} \\ &= 4r^2\{x^a, \phi\} + r^2\theta^{ba}\eth^b\phi \\ &= 3r^2\mathcal{A}^{(2)} .\end{aligned} \tag{5.13}$$

Similarly, one finds

$$\mathcal{I}(\mathcal{A}_a^{(1)}) := \{\theta^{ab}, \eth_b\phi\} = r^2\mathcal{A}_a^{(1)} \tag{5.14}$$

Now we can use the identities

$$\begin{aligned}\mathcal{I}(\mathcal{A}_a^{(2)}) &= \{\theta^{ab}, \theta^{bb'}\mathcal{A}_{b'}^{(1)}\} \\ &= \{\theta^{ab}, \theta^{bb'}\}\mathcal{A}_{b'}^{(1)} + \theta^{bb'}\{\theta^{ab}, \mathcal{A}_{b'}^{(1)}\} \\ &= 3r^2\theta^{ab'}\mathcal{A}_{b'}^{(1)} + \{\theta^{bb'}\theta^{ab}, \mathcal{A}_{b'}^{(1)}\} - \theta^{ab}\{\theta^{bb'}, \mathcal{A}_{b'}^{(1)}\} \\ &= 3r^2\theta^{ab'}\mathcal{A}_{b'}^{(1)} - r^2R^2\{P^{ab'}, \mathcal{A}_{b'}^{(1)}\} - \theta^{ab}\mathcal{I}(\mathcal{A}_b^{(1)}) \\ &= 3r^2\theta^{ab'}\mathcal{A}_{b'}^{(1)} - r^2(x^a\{x^c, \mathcal{A}_c^{(1)}\} - \theta^{ac}\mathcal{A}_c^{(1)}) - \theta^{ab}\mathcal{I}(\mathcal{A}_b^{(1)}) \\ &= 4r^2\theta^{ac}\mathcal{A}_c^{(1)} - \theta^{ab}\mathcal{I}(\mathcal{A}_b^{(1)}) ,\end{aligned} \tag{5.15}$$

wherein we used

$$\begin{aligned}R^2\{P^{ab}, \phi_b\} &= \{x^a x^c, \phi_c\} = x^a\{x^c, \phi_c\} + x^c\{x^a, \phi_c\} \\ &= x^a\{x^c, \phi_c\} - \theta^{ac}\phi_c ,\end{aligned} \tag{5.16}$$

for any tangential ϕ_c, and the gauge fixing relations (5.30). Therefore

$$\begin{aligned}\theta^{ab}\mathcal{I}(\mathcal{A}_b^{(1)}) &= 4r^2\mathcal{A}_a^{(2)} - \mathcal{I}(\mathcal{A}_a^{(2)}) \\ \tilde{\mathcal{I}}(\mathcal{A}_a^{(1)}) &= 4r^2\mathcal{A}_a^{(1)} + \frac{1}{r^2R^2}\theta^{ad}\mathcal{I}(\mathcal{A}_d^{(2)}) .\end{aligned} \tag{5.17}$$

For $s = 0$, this gives

$$\mathcal{I}(\mathcal{A}_a^{(2)}) = 3r^2\mathcal{A}^{(2)} \tag{5.18}$$

since $\mathcal{S}^2\mathcal{A}^{(2)} = 4\mathcal{A}^{(2)}$, in agreement with (5.13). Then (5.17) gives

$$\tilde{\mathcal{I}}(\mathcal{A}_a^{(1)}) = r^2\mathcal{A}_a^{(1)} \tag{5.19}$$

because $\mathcal{A}_a^{(1)}$ is tangential. To summarize,

$$\tilde{\mathcal{I}}\begin{pmatrix}\mathcal{A}_a^{(1)} \\ \mathcal{A}_a^{(2)}\end{pmatrix} = r^2\begin{pmatrix}1 & 0 \\ 0 & 3\end{pmatrix}\begin{pmatrix}\mathcal{A}_a^{(1)} \\ \mathcal{A}_a^{(2)}\end{pmatrix} \tag{5.20}$$

5.1.2. Spin 1 modes

Now let $\phi_a \in \mathcal{C}^0$ be a tangential, divergence-free spin 1 field. Then there are four tangential spin 1 modes, given by

$$\begin{aligned}\mathcal{A}_a^{(1)} &= \eth_a\phi^{(1)} \quad \in \mathcal{C}^1, \qquad \phi^{(1)} = \{x^a, \phi_a\} \in \mathcal{C}^1 \\ \mathcal{A}_a^{(2)} &= \theta^{ab}\eth_b\phi^{(1)} = \{x^a, \phi^{(1)}\} \quad \in \mathcal{C}^2 \oplus \mathcal{C}^0 , \\ \mathcal{A}_a^{(3)} &= \phi_a \quad \in \mathcal{C}^0 , \\ \mathcal{A}_a^{(4)} &= \theta^{ab}\phi_b \quad \in \mathcal{C}^1 .\end{aligned} \tag{5.21}$$

Here $\phi^{(1)}$ is the unique spin 1 mode in End($\mathcal{H}$). $\mathcal{I}$ can be computed on the $\mathcal{A}_a^{(3)}$ and $\mathcal{A}_a^{(4)}$ modes using

$$\mathcal{I}(\phi_a) := \{\theta^{ab}, \phi_b\} = r^2\phi_a \tag{5.22}$$

due to (D.2), which gives

$$\mathcal{I}(\mathcal{A}_a^{(3)}) := \{\theta^{ab}, \phi_b\} = r^2\phi_a = r^2\mathcal{A}_a^{(3)}. \tag{5.23}$$

Furthermore,

$$\begin{aligned}\tilde{\mathcal{I}}(\mathcal{A}_a^{(4)}) &:= P^{aa'}\{\theta^{a'b}, \theta^{bc}\phi_c\}\\ &= P^{aa'}\theta^{bc}\{\theta^{a'b}, \phi_c\} + P^{aa'}\{\theta^{a'b}, \theta^{bc}\}\phi_c\\ &= P^{aa'}\{\theta^{bc}\theta^{a'b}, \phi_c\} - \theta^{ab}\{\theta^{bc}, \phi_c\} + 3r^2\theta^{ac}\phi_c\\ &= -r^2R^2P^{aa'}\{P^{a'c}, \phi_c\} + 2r^2\theta^{ac}\phi_c\\ &= r^2P^{aa'}\theta^{a'c}\phi_c + 2r^2\mathcal{A}_a^{(4)}\\ &= 3r^2\mathcal{A}_a^{(4)}\end{aligned} \tag{5.24}$$

using (5.16). $\mathcal{I}(\mathcal{A}_a^{(2)})$ and $\mathcal{I}(\mathcal{A}_a^{(1)})$ will be computed for the general case below.

5.1.3. *Spin 2 modes*

Now let $\phi_{ab} = \phi_{ba} \in \mathcal{C}^0$ be a tangential, divergence-free, traceless spin 2 field, and let $\phi_a^{(2)} = \{x^b, \phi_{ab}\} \in \mathcal{C}^1$. Then there are four tangential spin 2 modes, given by

$$\begin{aligned}&\mathcal{A}_a^{(1)} = \eth_a\phi^{(2)} \ \in \mathcal{C}^2, \qquad \phi^{(2)} = \{x^a, \phi_a^{(2)}\} \in \mathcal{C}^2\,,\\ &\mathcal{A}_a^{(2)} = \theta^{ab}\eth_b\phi^{(2)} \ = \{x^a, \phi^{(2)}\} \quad \in \mathcal{C}^3 \oplus \mathcal{C}^1\,,\\ &\mathcal{A}_a^{(3)} = \phi_a^{(2)} \ \in \mathcal{C}^1\,,\\ &\mathcal{A}_a^{(4)} = \theta^{ab}\phi_b^{(2)} \ \in \mathcal{C}^2\,.\end{aligned} \tag{5.25}$$

Here $\phi^{(2)}$ is the unique spin 2 mode in End($\mathcal{H}$), which involves the linearized Riemann tensor. They satisfy the gauge fixing relations derived below, see (5.30). Also recall from (4.15) that $\phi_c^{(2)}$ is indeed tangential. Furthermore,

$$\begin{aligned}\mathcal{I}(\mathcal{A}_a^{(3)}) &:= \{\theta^{ab}, \phi_b^{(2)}\} = \{\theta^{ab}, \{x^c, \phi_{bc}\}\}\\ &= -\{\phi_{bc}, \{\theta^{ab}, x^c\}\} - \{x^c, \{\phi_{bc}, \theta^{ab}\}\}\\ &= r^2\{\phi_{bc}, \eta^{ac}x^b - \eta^{bc}x^a\} + r^2\{x^c, \phi_{ac}\}\\ &= 0\end{aligned} \tag{5.26}$$

using (5.22) for the last term. Adapting (5.27), we obtain

$$\begin{aligned}\tilde{\mathcal{I}}(\mathcal{A}_a^{(4)}) &= P^{aa'}\{\theta^{bc}\theta^{a'b}, \phi_c\} - \theta^{ab}\{\theta^{bc}, \phi_c\} + 3r^2\theta^{ac}\phi_c\\ &= -r^2R^2P^{aa'}\{P^{a'c}, \phi_c\} + 3r^2\theta^{ac}\phi_c\\ &= r^2P^{aa'}\theta^{a'c}\phi_c + 3r^2\mathcal{A}^{(4)}\\ &= 4r^2\mathcal{A}^{(4)}\,.\end{aligned} \tag{5.27}$$

It is illuminating to display the explicit tensor content of the spin 2 modes, recalling that $\phi_b^{(2)}$ is the spin connection (4.13) and $\phi^{(2)}$ is the curvature tensor. Using (4.17), this is

$$\begin{aligned}
\mathcal{A}_a^{(1)} &= \eth_a \phi^{(2)} = \frac{1}{2}\eth_a(\theta^{ed}\theta^{bc}\mathcal{R}_{ed;bc}[\phi^{(2)}])\,, \\
\mathcal{A}_a^{(2)} &= \frac{1}{2}\theta^{aa'}\eth_{a'}(\theta^{ed}\theta^{bc}\mathcal{R}_{ed;bc}[\phi^{(2)}])\,, \\
\mathcal{A}_a^{(3)} &= -\omega_{a;de}\theta^{de}\,, \\
\mathcal{A}_a^{(4)} &= -\theta^{ab}\omega_{b;de}\theta^{de}\,.
\end{aligned} \tag{5.28}$$

In particular, $\mathcal{A}_a^{(4)} = \theta^{ab}A_b$ encodes a $\mathfrak{so}(4,1)$-valued gauge field $A_b = -\omega_{b;de}\theta^{de}$ given by the linearized spin connection of ϕ_{ab}.

5.1.4. Spin $s \geq 1$ modes

Now consider the generic case. Let $\phi_{a_1\ldots a_s} \in \mathcal{C}^0$ be a tangential, divergence-free, traceless, symmetric spin s field, and let $\phi_a^{(s)} = \{x^{a_1}, \ldots \{x^{a_{s-1}}, \phi_{a_1\ldots a_{s-1}a}\}\ldots\} \in \mathcal{C}^{s-1}$. Then there are four tangential spin s modes, given by

$$\boxed{\begin{aligned}
&\mathcal{A}_a^{(1)} = \eth_a\phi^{(s)} \in \mathcal{C}^s, \qquad \phi^{(s)} = \{x^a, \phi_a^{(s)}\} \in \mathcal{C}^s \\
&\mathcal{A}_a^{(2)} = \theta^{ab}\eth_b\phi^{(s)} = \{x^a, \phi^{(s)}\} \quad \in \mathcal{C}^{s+1} \oplus \mathcal{C}^{s-1} \\
&\mathcal{A}_a^{(3)} = \phi_a^{(s)} \in \mathcal{C}^{s-1}\,, \\
&\mathcal{A}_a^{(4)} = \theta^{ab}\phi_b^{(s)} \in \mathcal{C}^s\,.
\end{aligned}} \tag{5.29}$$

Here $\phi^{(s)}$ is the unique spin s mode in End($\mathcal{H}$). The modes (5.29) satisfy the gauge-fixing relations

$$\begin{aligned}
\{x^a, \mathcal{A}_a^{(1)}\} &= \{x^a, \eth_a\phi^{(1)}\} = 0\,, \\
\{x^a, \mathcal{A}_a^{(4)}\} &= \{x^a, \theta^{ab}\phi_b\} = \theta^{ab}\{x^a, \phi_b\} = r^2R^2\eth^a\phi_a = 0\,, \\
\{x^a, \mathcal{A}_a^{(3)}\} &= \{x^a, \phi_a\} = \phi^{(1)}\,, \\
\{x^a, \mathcal{A}_a^{(2)}\} &= -\Box\phi^{(1)}\,,
\end{aligned} \tag{5.30}$$

using (D.3), and

$$\begin{aligned}
\eth^a\mathcal{A}_a^{(1)} &= -\frac{1}{r^2R^2}\Box\phi^{(1)}\,, \\
\eth^a\mathcal{A}_a^{(2)} &= 0 = \eth^a\mathcal{A}_a^{(3)}\,, \\
\eth^a\mathcal{A}_a^{(4)} &= \eth_a(\theta^{ab}\phi_b) = \{x^a, \phi_a\} = \phi^{(1)}\,.
\end{aligned} \tag{5.31}$$

These relations hold for any spin. Together with (5.7), it follows that $\mathcal{I}(\mathcal{A}^{(2)})$ and $\mathcal{I}(\mathcal{A}^{(3)})$ are tangential, while $\mathcal{I}(\mathcal{A}^{(1)})$ and $\mathcal{I}(\mathcal{A}^{(4)})$ are not. Let us proceed to $\tilde{\mathcal{I}}$; we first show that

$$\tilde{\mathcal{I}}(\mathcal{A}_a^{(3)}) := \{\theta^{ab}, \phi_b^{(s)}\} = (2-s)r^2\mathcal{A}_a^{(3)} \tag{5.32}$$

This can be proven inductively as follows:

$$\begin{aligned}
\{\theta^{ab}, \phi_b^{(s)}\} &= \{\theta^{ab}, \{x^c, \phi_{bc}^{(s)}\}\} \\
&= -\{\phi_{bc}^{(s)}, \{\theta^{ab}, x^c\}\} - \{x^c, \{\phi_{bc}^{(s)}, \theta^{ab}\}\} \\
&= r^2\{\phi_{ba}^{(s)}, x^b\} + (3-s)r^2\{x^c, \phi_{ac}^{(s)}\} \\
&= (2-s)r^2\phi_a^{(s)}
\end{aligned} \tag{5.33}$$

using (5.22), where $\phi^{(s)}_{ab} = \{x^{a_1}, \dots, \{x^{a_{s-2}}, \phi_{a_1\dots a_{s-2}ab}\}\dots\} \in \mathcal{C}^{s-2}$. Note that we employed the relation $\{\theta^{ab}, \phi^{(s)}_{bc}\} = (3-s)r^2\phi^{(s)}_{bc}$ for $\phi^{(s)}_{bc} \in \mathcal{C}^{s-2}$, which can be derived via induction, too. Adapting (5.27), this yields

$$\begin{aligned}\tilde{\mathcal{I}}(\mathcal{A}^{(4)}_a) &= P^{aa'}\{\theta^{bc}\theta^{a'b}, \phi_c\} - \theta^{ab}\{\theta^{bc}, \phi_c\} + 3r^2\theta^{ac}\phi_c \\ &= -r^2R^2P^{aa'}\{P^{a'c}, \phi_c\} - (2-s)r^2\theta^{ab}\phi^{(s)}_b + 3r^2\theta^{ac}\phi_c \\ &= r^2P^{aa'}\theta^{a'c}\phi_c + (s+1)\phi^{(s)}_a r^2\mathcal{A}^{(4)} \\ &= (s+2)r^2\mathcal{A}^{(4)}.\end{aligned} \tag{5.34}$$

To compute $\mathcal{I}(\mathcal{A}^{(2)}_a)$, consider

$$\begin{aligned}\mathcal{I}(\mathcal{A}^{(2)}_a) &:= \{\theta^{ab}, \mathcal{A}^{(2)}_b\} = \{\{x^a, x^b\}, \mathcal{A}^{(2)}_b\} \\ &= -\{\{x^b, \mathcal{A}^{(2)}_b\}, x_a\} - \{\{\mathcal{A}^{(2)}_b, x_a\}, x_b\},\end{aligned} \tag{5.35}$$

where the second term can be rewritten as

$$\begin{aligned}-\{\{\mathcal{A}^{(2)}_b, x_a\}, x_b\} &= -\{\{\{x_b, \phi\}, x_a\}, x_b\} \\ &= \{\{\{\phi, x_a\}, x_b\}, x_b\} + \{\{\theta^{ab}, \phi\}, x_b\} \\ &= \Box\mathcal{A}^{(2)}_a - \{\{\phi, x_b\}, \theta^{ab}\} - \{\{x_b, \theta^{ab}\}, \phi\} \\ &= \Box\mathcal{A}^{(2)}_a - \mathcal{I}(\mathcal{A}^{(2)}_b) + 4r^2\mathcal{A}^{(2)}_a.\end{aligned} \tag{5.36}$$

So that we obtain

$$\begin{aligned}2\mathcal{I}(\mathcal{A}^{(2)}_a) &= -\{\{x^b, \mathcal{A}^{(2)}_b\}, x_a\} + \Box\mathcal{A}^{(2)}_a + 4r^2\mathcal{A}^{(2)}_a \\ &= \{\Box\phi, x_a\} + \left(\Box + 4r^2\right)\mathcal{A}^{(2)}_a\end{aligned} \tag{5.37}$$

for any spin $s \geq 1$. Therefore

$$(\Box - 2\mathcal{I})(\mathcal{A}^{(2)}_a) = -\{\Box\phi, x_a\} - 4r^2\mathcal{A}^{(2)}_a. \tag{5.38}$$

On the other hand, for a spin s field ϕ we have

$$\begin{aligned}\{x_a, \Box\phi\} &= \mathcal{A}^{(2)}[\Box\phi] = r^2\mathcal{A}^{(2)}[(-C^2 + \mathcal{S}^2)\phi] = r^2(2s(s+1) - C^2_{\text{full}})\mathcal{A}^{(2)}[\phi] \\ &= ((\Box + 2\mathcal{I}) - r^2\mathcal{S}^2 + 2r^2s(s+1) - 4r^2)\mathcal{A}^{(2)}[\phi],\end{aligned} \tag{5.39}$$

using the intertwiner property and (5.8), hence

$$\{x_a, \Box\phi\} - \Box\{x_a, \phi\} = (2\mathcal{I} - r^2\mathcal{S}^2 + 2r^2s(s+1) - 4r^2)\mathcal{A}^{(2)}[\phi]. \tag{5.40}$$

Comparing with (5.37), this gives

$$2\mathcal{I}(\mathcal{A}^{(2)}_a) = -\{x^a, \Box\phi\} + \left(\Box + 4r^2\right)\mathcal{A}^{(2)}_a = (4r^2 - 2\mathcal{I} + r^2\mathcal{S}^2 - 2r^2s(s+1) + 4r^2)\mathcal{A}^{(2)}$$

such that

$$\boxed{2\mathcal{I}(\mathcal{A}^{(2)}_a) = r^2\left(\frac{1}{2}\mathcal{S}^2 - s(s+1) + 4\right)\mathcal{A}^{(2)}} \tag{5.41}$$

for $s \geq 1$, which is tangential. Hence if $\mathcal{S}^2$ is diagonal then $\mathcal{I}$ is also diagonal, and the Casimir $C^2[SO(4,1)]$ (5.8) can be diagonalized simultaneously. To evaluate (5.41), we decompose $\mathcal{A}^{(2)}_a$ into its components in $\mathcal{C}^{s-1} \oplus \mathcal{C}^{s+1}$ as follows:

$$\begin{aligned} \mathcal{A}_a^{(2)} &= -\alpha_s(\Box - 2r^2)\mathcal{A}_a^{(3)} + \mathcal{A}_a^{(2')} &&\in \mathcal{C}^{s-1} \oplus \mathcal{C}^{s+1} , \\ \mathcal{A}_a^{(2')} &:= \mathcal{A}_a^{(2)} + \alpha_s(\Box - 2r^2)\mathcal{A}_a^{(3)} &&\in \mathcal{C}^{s+1} , \end{aligned} \tag{5.42}$$

using (4.29); recall that $\mathcal{A}_a^{(3)} \equiv \phi_a^{(s)}$. Note that (5.42) is simultaneously a decomposition into eigenvectors of $\mathcal{I}$,

$$\begin{aligned} 2\mathcal{I}(\mathcal{A}_a^{(2')}) &= 2(s+3)r^2\mathcal{A}_a^{(2')}, \\ 2\mathcal{I}(\mathcal{A}_a^{(3)}) &= 2(2-s)r^2\mathcal{A}_a^{(3)} \end{aligned} \tag{5.43}$$

consistent with (5.32). Then we arrive at

$$\begin{aligned} 2\mathcal{I}(\mathcal{A}_a^{(2)}) &= -2(2-s)\alpha_s r^2(\Box - 2r^2)\mathcal{A}_a^{(3)} + 2(s+3)r^2\mathcal{A}_a^{(2')} \\ &= 2(s+3)r^2\mathcal{A}_a^{(2)} + 2(2s+1)\alpha_s r^2(\Box - 2r^2)\mathcal{A}_a^{(3)} \end{aligned} \tag{5.44}$$

and $\mathcal{I}(\mathcal{A}_a^{(1)})$ is obtained from (5.17),

$$\begin{aligned} \tilde{\mathcal{I}}(\mathcal{A}_a^{(1)}) &= 4r^2\mathcal{A}_a^{(1)} + \frac{1}{R^2}\theta^{ad}((s+3)\mathcal{A}_d^{(2)} + (2s+1)\alpha_s(\Box - 2r^2)\mathcal{A}_d^{(3)}) \\ &= r^2(1-s)\mathcal{A}_a^{(1)} + \frac{2s+1}{R^2}\alpha_s\theta^{ab}(\Box - 2r^2)\mathcal{A}_a^{(3)} \\ &= r^2(1-s)\mathcal{A}_a^{(1)} + \frac{2s+1}{R^2}\hat{\alpha}_s\mathcal{A}_a^{(4)} \end{aligned} \tag{5.45}$$

where the last line is only a short-hand notation which applies to irreps, cf. (4.10). Hence $\tilde{\mathcal{I}}$ is diagonalized as follows:

$$\begin{aligned} \tilde{\mathcal{I}}(\mathcal{A}_a^{(1')}) &= r^2(1-s)\mathcal{A}_a^{(1')}, \\ \mathcal{A}_a^{(1')} &= \mathcal{A}_a^{(1)} - \frac{\alpha_s}{R^2r^2}\tilde{\theta}^{ab}(\Box - 2r^2)\mathcal{A}_b^{(3)} \equiv \mathcal{A}_a^{(1)} - \frac{\hat{\alpha}_s}{R^2r^2}\mathcal{A}_a^{(4)} , \end{aligned} \tag{5.46}$$

using (5.34). Accordingly, we define the eigenmodes

$$(\mathcal{B}_a^{(1)}, \mathcal{B}_a^{(2)}, \mathcal{B}_a^{(3)}, \mathcal{B}_a^{(4)}) := (\mathcal{A}_a^{(1')}, \mathcal{A}_a^{(2')}, \mathcal{A}_a^{(3)}, \mathcal{A}_a^{(4)}) , \tag{5.47}$$

which satisfy

$$\tilde{\mathcal{I}}\begin{pmatrix}\mathcal{B}_a^{(1)}\\ \mathcal{B}_a^{(2)}\\ \mathcal{B}_a^{(3)}\\ \mathcal{B}_a^{(4)}\end{pmatrix} = r^2\begin{pmatrix}1-s & 0 & 0 & 0\\ 0 & s+3 & 0 & 0\\ 0 & 0 & 2-s & 0\\ 0 & 0 & 0 & 2+s\end{pmatrix}\begin{pmatrix}\mathcal{B}_a^{(1)}\\ \mathcal{B}_a^{(2)}\\ \mathcal{B}_a^{(3)}\\ \mathcal{B}_a^{(4)}\end{pmatrix} , \tag{5.48a}$$

$$\mathcal{S}^2\begin{pmatrix}\mathcal{B}_a^{(1)}\\ \mathcal{B}_a^{(2)}\\ \mathcal{B}_a^{(3)}\\ \mathcal{B}_a^{(4)}\end{pmatrix} = 2\begin{pmatrix}s(s+1) & 0 & 0 & 0\\ 0 & (s+1)(s+2) & 0 & 0\\ 0 & 0 & (s-1)s & 0\\ 0 & 0 & 0 & s(s+1)\end{pmatrix}\begin{pmatrix}\mathcal{B}_a^{(1)}\\ \mathcal{B}_a^{(2)}\\ \mathcal{B}_a^{(3)}\\ \mathcal{B}_a^{(4)}\end{pmatrix} . \tag{5.48b}$$

This shows that all these modes are distinct, and it will allow to diagonalize and evaluate explicitly the quadratic action. It also implies that we did not miss any modes, since there can be only 5 modes for each spin (including the radial one, see below).

Gauge fixing term. The intertwiner $\mathcal{G}$ of (5.6) takes the values

$$\begin{aligned}\mathcal{G}(\mathcal{A}_a^{(2)}) &= -\{x^a, \Box\phi^{(s)}\}\\ \mathcal{G}(\mathcal{A}_a^{(3)}) &= \{x^a, \phi^{(s)}\} = \mathcal{A}_a^{(2)}\\ \mathcal{G}(\mathcal{A}_a^{(1)}) &= \mathcal{G}(\mathcal{A}_a^{(4)}) = 0\,.\end{aligned} \tag{5.49}$$

5.2. Recombination, $\mathfrak{hs}$-valued gauge fields and Young diagrams

The distinct modes $\mathcal{A}^{(i)}$ are useful to disentangle the different degrees of freedom. On the other hand we can relax the requirements that the underlying tensor fields $\phi_{a_1\ldots a_s}$ are irreducible, so that the modes can be captured in a simpler way.

Trace contributions. These arise from

$$\phi_{a_1\ldots a_s} = \eta_{a_1a_2}\phi_{a_3\ldots a_s}\,. \tag{5.50}$$

Then

$$\begin{aligned}\tilde\phi_a^{(s)} &= \{x^{a_1},\ldots\{x^{a_{s-1}},\eta_{a_1a_2}\phi_{a_3\ldots a}\}\ldots\}\} = -\Box\phi_a^{(s-2)} \ \in \mathcal{C}^{s-2},\\ \tilde\phi^{(s)} &= \{x^a,\tilde\phi_a^{(s)}\} \ = -\{x^a,\Box\phi_a^{(s)}\}\end{aligned} \tag{5.51}$$

which enters the four modes as follows

$$\begin{aligned}\mathcal{A}_a^{(1)} &= \eth_a\tilde\phi^{(s-2)} \ \in \mathcal{C}^{s-2},\\ \mathcal{A}_a^{(2)} &= \theta^{ab}\eth_b\tilde\phi^{(s-2)} \ = \{x^a,\phi^{(s-2)}\} \quad \in \mathcal{C}^{s-1}\oplus\mathcal{C}^{s-3}\\ \mathcal{A}_a^{(3)} &= \tilde\phi_a^{(s-2)} \ \in \mathcal{C}^{s-3},\\ \mathcal{A}_a^{(4)} &= \theta^{ab}\tilde\phi_b^{(s-2)} \ \in \mathcal{C}^{s-2}.\end{aligned} \tag{5.52}$$

Hence the trace components reproduce the four modes with spin $s-2$, as long as $\Box\phi_a^{(s)} \neq 0$.

Divergence modes. Now we drop the requirement that ϕ_{ab} is divergence-free. Consider the case of rank 2 tensors, expressed in terms of spin 1 modes as in (4.22)

$$\tilde\phi_{ab}^{(1)} = \nabla_a\phi_b + \nabla_b\phi_a\,. \tag{5.53}$$

Then according to (4.24), these contributions to the would-be spin 2 modes $\mathcal{A}_a^{(1)}, \mathcal{A}_a^{(2)}$ vanish identically. The contribution to $\mathcal{A}_a^{(3)}$ reduces to a combinations of the spin 1 modes of $\mathcal{A}_a^{(1)}$ and $\mathcal{A}_a^{(4)}$, and the contribution to $\mathcal{A}_a^{(4)}$ reduces to a combinations of the spin 1 modes of $\mathcal{A}_a^{(2)}$ and $\mathcal{A}_a^{(3)}$. Hence if we drop the divergence-free condition, it would suffice to keep the $\mathcal{A}_a^{(3)}$ and $\mathcal{A}_a^{(4)}$ modes.[14] In particular, we need not worry about these constraints upon projecting H^4 to $\mathcal{M}^{3,1}$. It will suffice to impose the appropriate divergence- and trace-conditions for $\mathcal{M}^{3,1}$.

Finally as for S_N^4 [2], we can collect *all* tangential fluctuation modes as $\mathfrak{hs}$-valued tangential gauge fields

$$\mathcal{A}^a = \theta^{ac}\mathbf{A}_c, \qquad \mathbf{A}_c = A_{c,\underline{\alpha}}(x)\,\Xi^{\underline{\alpha}} \tag{5.54}$$

[14] However, the spin 0 modes cannot be recovered from divergence modes: for $\phi_a = \eth_a\phi$ we get $\tilde\phi^{(1)} = \{x^a, \eth_a\phi\} = 0$ due to (D.3).

where $A_{c,\underline{\alpha}}(x)$ are double-traceless tensor fields corresponding to 2-row Young diagrams of the type $\begin{smallmatrix}\Box&\Box&\Box\\ \Box&\Box&\end{smallmatrix}$. The external leg is associated to the extra box in the Young diagram. However the $A_{c,\underline{\alpha}}(x)$ are in fact higher curvatures of the underlying symmetric tensor fields $\phi_{a_1\ldots a_s}$ as in (4.28), which characterize the irreducible physical degrees of freedom $\mathcal{A}_a^{(i)}$.

5.2.1. Inner products

The inner products (5.5) of the tangential fluctuations are given by

$$\begin{aligned}
\int \mathcal{A}_b^{(1)}\mathcal{A}_b^{(1)} &= \int \eth_b\phi^{(s)}\eth_b\phi^{(s)} = \frac{\alpha_s}{r^2R^2}\int \phi_a^{(s)}(\Box+2r^2 s)(\Box-2r^2)\phi_a^{(s)}\,,\\
\int \mathcal{A}_b^{(1)}\mathcal{A}_b^{(4)} &= \int \eth_a\phi^{(s)}\theta^{ab}\phi_b^{(s)} = \int \phi^{(s)}\{x_b,\phi_b^{(s)}\}\\
&= \int \phi^{(s,1)}\phi^{(s,4)} = \alpha_s\int \phi_a^{(s)}(\Box-2r^2)\phi_a^{(s)}\,,\\
\int \mathcal{A}_b^{(3)}\mathcal{A}_b^{(2)} &= \int \phi_b^{(s)}\{x^b,\phi^{(s)}\} = -\int \{x^b,\phi_b^{(s)}\}\phi^{(s)} = -\alpha_s\int \phi_a^{(s)}(\Box-2r^2)\phi_a^{(s)}\,,\\
\int \mathcal{A}_b^{(2)}\mathcal{A}_b^{(2)} &= \int \{x_b,\phi^{(s)}\}\{x^b,\phi^{(s)}\} = \alpha_s\int \phi_a^{(s)}(\Box+2r^2 s)(\Box-2r^2)\phi_a^{(s)}\,,\\
\int \mathcal{A}_a^{(3)}\mathcal{A}_a^{(3)} &= \int \phi_b^{(s)}\phi_b^{(s)}\,,\\
\int \mathcal{A}_a^{(4)}\mathcal{A}_a^{(4)} &= \int \theta^{ab}\phi_b^{(s)}\theta^{ac}\phi_c^{(s)} = r^2R^2\int \phi_b^{(s)}\phi_b^{(s)}\,,\\
\int \mathcal{A}_b^{(1)}\mathcal{A}_b^{(2)} &= \int \mathcal{A}_b^{(1)}\mathcal{A}_b^{(3)} = \int \mathcal{A}_a^{(4)}\mathcal{A}_a^{(2)} = \int \mathcal{A}_a^{(4)}\mathcal{A}_a^{(3)} = 0\,,
\end{aligned}\tag{5.55}$$

using (3.69), (D.3), (D.36) and $[\theta^{ab}\phi_b^{(s,4)}\phi_a^{(s,3)}]_0 = 0$; we drop the labels $\phi_b^{(s,4)} \equiv \phi_b^{(s)}$ if no confusion can arise.

Now consider the eigenstates (5.47) of $\mathcal{I}$. We verify that $\mathcal{B}_a^{(2)}$ and $\mathcal{B}_a^{(1)}$ satisfy the orthogonality relations

$$\begin{aligned}
\int \mathcal{B}_a^{(2)}\mathcal{B}_a^{(3)} &= \int (\mathcal{A}_a^{(2)}+\hat{\alpha}_s\mathcal{A}_a^{(3)})\mathcal{A}_a^{(3)} = 0\,,\\
\int \mathcal{B}_a^{(1)}\mathcal{B}_a^{(4)} &= 0\,,
\end{aligned}\tag{5.56}$$

using the definitions (5.42), (5.46) as well as (4.44). Therefore $\{\mathcal{B}_a^{(i)}\}_{i=1}^4$ form an orthogonal basis of eigenmodes. The normalization can be computed as

$$\begin{aligned}
\int \mathcal{B}_a^{(2)}\mathcal{B}_a^{(2)} &= \int \left(\mathcal{A}_a^{(2)}+\hat{\alpha}_s\mathcal{A}_a^{(3)}\right)\left(\mathcal{A}_a^{(2)}+\hat{\alpha}_s\mathcal{A}_a^{(3)}\right)\\
&= \alpha_s\int \phi_a^{(2')}\Big((\Box+2r^2 s)-\alpha_s(\Box-2r^2)\Big)(\Box-2r^2)\phi_a^{(2')}\\
&= \int \phi^{(2')}\Big((1-\alpha_s)\Box+2r^2(s+1)\Big)\phi^{(2')}
\end{aligned}\tag{5.57}$$

and

$$\begin{aligned}\int \mathcal{B}_a^{(1)}\mathcal{B}_a^{(1)} &= \int \left(\mathcal{A}_a^{(1)} - \frac{\hat{\alpha}_s}{R^2r^2}\mathcal{A}_a^{(4)}\right)\left(\mathcal{A}_a^{(1)} - \frac{\hat{\alpha}_s}{R^2r^2}\mathcal{A}_a^{(4)}\right) \\ &= \frac{\alpha_s}{r^2R^2}\int \phi_a^{(1')}\Big((\Box + 2r^2 s) - \alpha_s(\Box - 2r^2)\Big)(\Box - 2r^2)\phi_a^{(1')} \\ &= \frac{1}{r^2R^2}\int \phi^{(1')}\Big((1-\alpha_s)\Box + 2r^2\alpha_s(s+1)\Big)\phi^{(1')} \,. \end{aligned} \tag{5.58}$$

Note that all $\mathcal{A}_a^{(2)}$ modes are pure gauge modes, and they will drop out in the action.

5.3. *Radial modes*

Finally consider the radial fluctuation modes. These are given by

$$\mathcal{A}_a^{(r)}[\phi^{(s)}] = x_a\phi^{(s)}, \qquad \phi^{(s)} \in \mathcal{C}^s \,. \tag{5.59}$$

They are dangerous because the radial metric in $\mathbb{R}^{1,4}$ is negative,

$$\int \mathcal{A}_b^{(r)}\mathcal{A}_b^{(r)} = \int x_a\phi^{(s)}x^a\phi^{(s)} = -R^2\int \phi^{(s)}\phi^{(s)} \tag{5.60}$$

recalling that $x_a x^a = -R^2 < 0$. However, they disappear after the projection to $\mathcal{M}^{3,1}$. If we include these radial fluctuations, we should first diagonalize $\mathcal{I}$. We have

$$\begin{aligned}\mathcal{I}(\mathcal{A}_a^{(r)}) &= \{\theta^{ab}, x_b\phi^{(s)}\} = \{\theta^{ab}, x_b\}\phi^{(s)} + x_b\{\theta^{ab}, \phi^{(s)}\} \\ &= 4x^a\phi^{(s)} - \theta^{ab}\{x_b, \phi^{(s)}\} \\ &= 4\mathcal{A}_a^{(r)} + r^2R^2\mathcal{A}_a^{(1)} \,. \end{aligned} \tag{5.61}$$

Recall that $\mathcal{I}(\mathcal{A}^{(2',3)})$ is tangential, but $\mathcal{I}(\mathcal{A}^{(1,4)})$ is not, with

$$\begin{aligned} x^a\mathcal{I}(\mathcal{A}^{(1)}[\phi]) &= \Box\phi \,, \\ x^a\mathcal{I}(\mathcal{A}^{(4)}[\phi]) &= -r^2R^2\phi \,, \\ x^a\mathcal{I}(\mathcal{A}^{(1')}[\phi]) &= (\Box + \hat{\alpha}_s)\phi \,, \end{aligned} \tag{5.62}$$

using (5.7) and (5.31). Hence the radial modes may couple to the $\mathcal{A}^{(1,2)}$ or the $\mathcal{B}^{(1,2)}$ modes, and the $\mathcal{I}$ eigenmodes seem to mix completely all 3 components $\mathcal{A}^{(1')}$, $\mathcal{A}^{(4)}$, $\mathcal{A}^{(r)}$. However since the radial modes are negative definite, we will focus on the tangential modes, and on its projection to $\mathcal{M}^{4,1}$ in the next stage.

5.4. $SO(4,1)$*-invariant quadratic action on* H^4

The quadratic fluctuations for the fluctuation modes $y^a = x^a + \mathcal{A}^a$ are governed by the action

$$S[y] = S[x] + S_2[\mathcal{A}] + O(\mathcal{A}^3), \tag{5.63}$$

where

$$S_2[\mathcal{A}] = \frac{2}{g^2}\int d\mu\Big(\mathcal{A}_a(\mathcal{D}^2\mathcal{A})^a + \{x^a, \mathcal{A}_a\}^2\Big) = \frac{2}{g^2}\int d\mu\, \mathcal{A}_a(\mathcal{D}^2 + \mathcal{G})\mathcal{A}^a \,. \tag{5.64}$$

Here

$$(\mathcal{D}^2\mathcal{A}) := \left(\Box - 2\mathcal{I} + \frac{1}{2}\mu^2\right)\mathcal{A} \tag{5.65}$$

is the "vector" (matrix) Laplacian, and $\mathcal{G}(\mathcal{A})$ (5.6) ensures gauge invariance. The mass term determines r^2 via the on-shell condition for H^4_n,

$$0 = \left(\Box + \frac{1}{2}\mu^2\right)x^a, \qquad \frac{1}{2}\mu^2 = 4r^2 \ . \tag{5.66}$$

Gauge-invariant action. Consider first the gauge-invariant kinetic term

$$S_2[\mathcal{A}] = \frac{2}{g^2}\int d\mu\, \mathcal{A}_a(\mathcal{D}^2 + \mathcal{G})\mathcal{A}^a \ . \tag{5.67}$$

We verify that the pure gauge modes $\mathcal{A}^{(2)}_a$ are null modes using (5.49) and (5.38):

$$(\mathcal{D}^2 + \mathcal{G})\mathcal{A}^{(2)}_a = -\{\Box\phi^{(s)}, x_a\} + \left(\frac{1}{2}\mu^2 - 4r^2\right)\mathcal{A}^{(2)}_a - \{x^a, \Box\phi^{(s)}\} = 0 \tag{5.68}$$

for any spin, taking into account the on-shell condition $\frac{1}{2}\mu^2 = 4r^2$. Hence the pure gauge modes $\mathcal{A}^{(2)}_a$ indeed decouple.

For spin 0, we determine the action explicitly for the $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ modes

$$(\mathcal{D}^2 + \mathcal{G})\begin{pmatrix}\mathcal{B}^{(1)}_a \\ \mathcal{B}^{(2)}_a\end{pmatrix} = \begin{pmatrix}\Box + 2r^2 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathcal{B}^{(1)}_a \\ \mathcal{B}^{(2)}_a\end{pmatrix} \ . \tag{5.69}$$

The inner product is diagonal for spin 0, and the quadratic action is given by

$$S_2[\mathcal{A}] = \int \mathcal{B}^{(i)}_a \begin{pmatrix}\Box + 2r^2 & 0 \\ 0 & 0\end{pmatrix}\mathcal{B}^{(i)}_a \ . \tag{5.70}$$

Since $\mathcal{B}^{(1)}_a \in \mathcal{C}^0$, this is indeed positive define (except for the pure gauge mode) due to (5.55), recalling that $\Box \propto -\eth\cdot\eth$ for spin 0 (3.69).

Gauge-fixed action and positivity. Now we consider a gauge-fixed action, which is obtained by canceling $\mathcal{G}$ with a suitable Faddeev–Popov (or BRST) term:

$$S_{2,(\mathrm{fix})}[\mathcal{A}] = \frac{2}{g^2}\int d\mu\, \mathcal{A}\mathcal{D}^2\,\mathcal{A} \ . \tag{5.71}$$

We work in the basis $\{\mathcal{B}^{(i)}\}$ (5.47) where $\mathcal{I}$ is diagonal. Then the eigenvalues of the kinetic operator $\mathcal{D}^2$ are elaborated in the appendix D.1. Together with the inner products in section 5.2.1, we obtain the following diagonalized quadratic action

$$\begin{aligned}\int \mathcal{B}^{(1)}_a\mathcal{D}^2\mathcal{B}^{(1)}_a &= \frac{\alpha_s}{r^2R^2}\int \phi^{(s)}_a \\ &\quad \times \Big((\Box + 2r^2 s) - \alpha_s(\Box - 2r^2)\Big)(\Box - 2r^2)(\Box + 2r^2(3s+2))\phi^{(s)}_a \\ &= \frac{1}{r^2R^2}\int \phi^{(s)}\Big((1-\alpha_s)\Box + 2r^2\alpha_s(s+1)\Big)(\Box + 4r^2(s+1))\phi^{(s)} \ ,\end{aligned} \tag{5.72a}$$

$$\int \mathcal{B}_a^{(2)} \mathcal{D}^2 \mathcal{B}_a^{(2)} = \alpha_s \int \phi_a^{(s)} \Big((\Box + 2r^2 s) - \alpha_s (\Box - 2r^2) \Big) (\Box - 2r^2)(\Box + 2r^2 s) \phi_a^{(s)}$$
$$= \int \phi^{(s)} \Big((1-\alpha_s)\Box + 2r^2 \alpha_s (s+1) \Big) \Box \phi^{(s)} , \tag{5.72b}$$
$$\int \mathcal{B}_a^{(3)} \mathcal{D}^2 \mathcal{B}_a^{(3)} = \int \phi_b^{(s)} (\Box + 2r^2 s) \phi_b^{(s)} , \tag{5.72c}$$
$$\int \mathcal{B}_a^{(4)} \mathcal{D}^2 \mathcal{B}_a^{(4)} = r^2 R^2 \int \phi_b^{(s)} (\Box - 2r^2 s) \phi_b^{(s)} . \tag{5.72d}$$

All these terms are non-negative, because

$$(\Box + 2r^2 s) - \alpha_s (\Box - 2r^2) = (1-\alpha_s)\Box + 2r^2 (s + \alpha_s) > 0.$$
$$\mathcal{A}[(\Box - 2r^2 s)\phi_b^{(s)}] \propto \mathcal{A}[(\Box + r^2 s(s-3))\phi_{a_1 \ldots a_s}] \tag{5.73}$$

for any intertwiner $\mathcal{A}$, using (D.33). The first line is positive because $1 > \alpha_s$, and the second line is positive since $\Box + r^2 s(s-3)$ is manifestly positive for $s \geq 3$, while for $s = 1, 2$ it coincides with $\Box - 2r^2$ which is also positive on divergence-free tensor fields as shown in (4.39). As usual, the unphysical modes will be canceled by Faddeev–Popov ghosts.

We consider explicitly the case of spin 1 and spin 2. For spin 1, we have

$$\int \mathcal{B}_a^{(1)} \mathcal{D}^2 \mathcal{B}_a^{(1)} = \frac{\alpha_1}{r^2 R^2} \int \phi_a \Big((\Box + 2r^2) - \alpha_1 (\Box - 2r^2) \Big) (\Box - 2r^2)(\Box + 10r^2) \phi_a$$
$$= \frac{1}{r^2 R^2} \int \phi^{(1)} \Big((1-\alpha_1)\Box + 4r^2 \alpha_1 \Big) (\Box + 8r^2) \phi^{(1)} , \tag{5.74a}$$
$$\int \mathcal{B}_a^{(2)} \mathcal{D}^2 \mathcal{B}_a^{(2)} = \alpha_1 \int \phi_a \Big((\Box + 2r^2) - \alpha_1 (\Box - 2r^2) \Big) (\Box - 2r^2)(\Box + 2r^2) \phi_a$$
$$= \int \phi^{(1)} \Big((1-\alpha_1)\Box + 4r^2 \alpha_1 \Big) \Box \phi^{(1)} , \tag{5.74b}$$
$$\int \mathcal{B}_a^{(3)} \mathcal{D}^2 \mathcal{B}_a^{(3)} = \int \phi_a (\Box + 2r^2) \phi_a , \tag{5.74c}$$
$$\int \mathcal{B}_a^{(4)} \mathcal{D}^2 \mathcal{B}_a^{(4)} = r^2 R^2 \int \phi_a (\Box - 2r^2) \phi_a , \tag{5.74d}$$

and for spin 2

$$\int \mathcal{B}_a^{(1)} \mathcal{D}^2 \mathcal{B}_a^{(1)} = \frac{\alpha_2}{r^2 R^2} \int \phi_a^{(2)} \Big((\Box + 4r^2) - \alpha_2 (\Box - 2r^2) \Big) (\Box - 2r^2)(\Box + 16r^2) \phi_a^{(2)}$$
$$= \frac{1}{r^2 R^2} \int \phi^{(2)} \Big((1-\alpha_2)\Box + 6r^2 \alpha_2 \Big) (\Box + 12r^2) \phi^{(2)}$$
$$= \frac{\alpha_1 \alpha_2}{r^2 R^2} \int \phi_{ab} (\Box + 6r^2 - \alpha_2 \Box)(\Box + 18r^2)(\Box - 2r^2) \Box \phi_{ab} , \tag{5.75a}$$
$$\int \mathcal{B}_a^{(2)} \mathcal{D}^2 \mathcal{B}_a^{(2)} = \alpha_2 \int \phi_a^{(2)} \Big((\Box + 4r^2) - \alpha_2 (\Box - 2r^2) \Big) (\Box - 2r^2)(\Box + 4r^2) \phi_a^{(2)}$$
$$= \int \phi^{(2)} \Big((1-\alpha_2)\Box + 6r^2 \alpha_2 \Big) \Box \phi^{(2)}$$
$$= \alpha_2 \alpha_1 \int \phi_{ab} \Big(\Box + 6r^2 - \alpha_2 \Box \Big) (\Box + 6r^2)(\Box - 2r^2) \Box \phi_{ab} , \tag{5.75b}$$

$$\begin{aligned}\int \mathcal{B}_a^{(3)} \mathcal{D}^2 \mathcal{B}_a^{(3)} &= \int \phi_a^{(2)} (\Box + 4r^2) \phi_a^{(2)} \\ &= \alpha_1 \int \phi_{ab} (\Box + 6r^2)(\Box - 2r^2) \phi_{ab} \,, \end{aligned} \tag{5.75c}$$

$$\begin{aligned}\int \mathcal{B}_a^{(4)} \mathcal{D}^2 \mathcal{B}_a^{(4)} &= r^2 R^2 \int \phi_b^{(2)} (\Box - 4r^2) \phi_b^{(2)} \\ &= \alpha_1 r^2 R^2 \int \phi_{ab} (\Box - 2r^2)^2 \phi_{ab} \,, \end{aligned} \tag{5.75d}$$

using (4.43). Note that we only include tangential fluctuation modes here. If we would also include the radial fluctuations as in section 5.3, they would be negative definite or ghost modes, because the metric in the radial direction is time-like. However this is resolved upon projecting to $\mathcal{M}^{3,1}$, as discussed below.

5.5. Yang–Mills gauge theory

We can write the full action (5.1) in a conventional (higher-spin) Yang–Mills form for the recombined higher-spin gauge fields (5.54) $\mathcal{A}^a = \theta^{ab} \mathbf{A}_b$. Then the field strength is

$$\begin{aligned}\mathcal{F}^{ab} &= [X^a + \mathcal{A}^a, X^b + \mathcal{A}^b] \sim \theta^{ab} + \theta^{aa'} \theta^{bb'} F_{a'b'}, \\ F_{ab} &= \nabla_a \mathbf{A}_b - \nabla_b \mathbf{A}_{a'} + [\mathbf{A}_a, \mathbf{A}_b] \end{aligned} \tag{5.76}$$

recalling that $\nabla \theta^{ab} = 0$. Hence the action (5.1)

$$S[Y] \sim \frac{1}{g_{YM}^2} \int\limits_{H^4} \left(F_{ab} F_{a'b'} \eta^{aa'} \eta^{bb'} - \frac{2}{R^2} \mathbf{A}_a \mathbf{A}_{a'} \eta^{aa'} \right) \tag{5.77}$$

is basically a $\mathfrak{hs}$-valued Yang–Mills action[15] (dropping surface terms and using $\mu^2 = 8r^2$), where

$$\frac{1}{g_{\mathrm{YM}}^2} = \rho \, \frac{L_{NC}^8}{4g^2} \tag{5.78}$$

is the dimensionless Yang–Mills coupling constant. For nonabelian spin 1 modes $\mathcal{A}_a^{(4)}$ on stacks of H_n^4 branes, the usual Yang–Mills action is recovered. For spin 2, one would expect this to describe some type of quadratic gravity action [32–34]. However this does not happen as shown below, since the graviton is obtained by a field redefinition (5.101) and does not propagate at the classical level. However the Yang–Mills framework suggests that no ghost modes appear also for higher-spin (as opposed to quadratic gravity), hence gravity might emerge at the quantum level.

5.6. Metric and gravitons on H^4

Now we take some of the leading (cubic) interactions of these modes into account, focusing on the contributions of the spin 2 (and spin 1) modes to the kinetic term on H^4. These contributions

[15] We used $x_a \mathcal{A}^a = 0$; the apparent “mass” term is at the cosmological curvature scale, and would presumably disappear upon imposing the non-linear constraint $Y_a Y^a = -R^2$.

are expected to give rise to linearized gravity. The kinetic term for all fluctuations on a given background $Y^a \sim y^a$ arises in the matrix model from[16]

$$\begin{aligned} S[\phi] = -\mathrm{Tr}[Y^a,\phi][Y_a,\phi] &\sim \int \rho\,\{y^a,\phi\}\{y_a,\phi\} \\ &= \int \rho\gamma^{ab}\eth_a\phi\eth_b\phi \stackrel{\xi}{=} \int \rho\,\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= \int\limits_{H^4} d^4x\,\sqrt{|G_{\mu\nu}|}\,G^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \end{aligned} \tag{5.79}$$

using (3.77); some dimensionful constants are absorbed in φ, and Greek indices indicate local coordinates. Here γ^{ab} is a symmetric tensor in $SO(4,1)$ notation

$$\gamma^{ab} = \eta_{cc'}e^{ca}e^{c'b}, \qquad e^{ca} = \{y^c, x^a\} \tag{5.80}$$

which in local coordinates near some reference point ξ reduces to $\gamma^{\mu\nu}$, cf. (3.58d). Hence the effective metric is given by [3,15,19]

$$G^{\mu\nu} = \frac{4\alpha}{L_{NC}^4}\,\gamma^{\mu\nu}\,, \qquad \alpha = \sqrt{\frac{L_{NC}^4}{4|\gamma^{\mu\nu}|}} \tag{5.81}$$

and e^{ca} can be interpreted as vielbein. For a deformation of the H^4 background of the form

$$y^a = x^a + \mathcal{A}^a\,, \tag{5.82}$$

the metric is perturbed due to $\gamma^{ab} = \overline{\gamma}^{ab} + \delta_{\mathcal{A}}\gamma^{ab} + O(\mathcal{A}^2)$ with

$$\begin{aligned} \delta_{\mathcal{A}}\gamma^{ab} =: H^{ab}[\mathcal{A}] &= \{x^c, x^a\}\{\mathcal{A}_c, x^b\} + (a \leftrightarrow b) \\ &= \theta^{ca}\{\mathcal{A}_c, x^b\} + (a \leftrightarrow b) \\ &= \{\theta^{ca}\mathcal{A}_c, x^b\} + \{\theta^{cb}\mathcal{A}_c, x^a\} + r^2\left(\mathcal{A}^b x^a + \mathcal{A}^a x^b - 2\eta^{ab}\left(\mathcal{A}_c x^c\right)\right)\,. \end{aligned} \tag{5.83}$$

Here $H^{ab}[\mathcal{A}]$ is an $SO(4,1)$ intertwiner and tangential,

$$H^{ab}x_a = 0\,, \qquad H := \eta_{ab}H^{ab} = \frac{1}{2}L_{NC}^4\eth^a\mathcal{A}_a\,. \tag{5.84}$$

Then the linearized effective metric (5.81) becomes in $SO(4,1)$-covariant notation

$$G^{ab} = P^{ab} + \widetilde{h}^{ab}\,, \qquad \text{with} \qquad \widetilde{h}^{ab} := \left(h^{ab} - \frac{1}{2}P^{ab}h\right)\,, \tag{5.85}$$

thus defining the *physical graviton* $\widetilde{h}^{ab}$, where

$$h^{ab} = \frac{4}{L_{NC}^4}[H^{ab}]_0, \qquad h = \eta_{ab}h^{ab} \tag{5.86}$$

is dimensionless. We study the graviton modes (5.85) for the spin $s = 0, 1, 2$ fluctuations of (5.29) in more detail below.

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[16] One might worry about the contributions from $\{y^a,\cdot\}$ on the generators θ^{bc} for higher-spin modes. However the metric is always defined by the two derivative terms acting on the tensor fields.

5.6.1. Spin 0 gravitons

To begin with, consider the perturbation (5.83) of the metric for the two spin 0 modes of (5.11). One finds

$$H_{ab}[\mathcal{A}^{(1)}] = \theta^{ac}\theta^{bd}\left(\eth_c\eth_d\phi^{(0)} + \eth_d\eth_c\phi^{(0)}\right),$$
$$H_{ab}[\mathcal{A}^{(2)}] = r^2(x_a\theta^{bd}\eth_d\phi^{(0)} - R^2\theta^{ad}\eth_d\eth_b\phi^{(0)}) + (a \leftrightarrow b)\,. \tag{5.87}$$

Upon averaging, one obtains

$$h_{ab}[\mathcal{B}^{(1)}] = \alpha_1\left(2P_{ab}\eth\cdot\eth\phi^{(0)} - (\nabla_a\nabla_b\phi^{(0)} + \nabla_b\nabla_a\phi^{(0)})\right), \tag{5.88}$$
$$h_{ab}[\mathcal{B}^{(2)}] = 0\,,$$

and the expressions satisfy

$$h[\mathcal{B}^{(1)}] = 6\Box\phi, \qquad \nabla^a h_{ab}[\mathcal{B}^{(1)}] = 0\,, \tag{5.89}$$

using $\nabla_a h^{ab} = \eth_a h^{ab} - \frac{1}{R^2}x_b h$. Then the physical graviton of (5.85) satisfies the de Donder gauge,

$$\nabla^a\tilde{h}_{ab}[\mathcal{B}^{(1)}] - \frac{1}{2}\nabla_b\tilde{h} = 0 \qquad \text{with} \qquad \widetilde{h}_{ab}[\mathcal{B}^{(1)}] = h_{ab}[\mathcal{B}^{(1)}] - \frac{1}{2}P_{ab}\tilde{h}\,. \tag{5.90}$$

The spin 0 contribution to the metric is interesting because its off-shell modes have the wrong (ghost-like) sign in GR. This does not happen in the present Yang–Mills model, which is important for quantization.

5.6.2. Spin 1 gravitons

Next, we compute the spin one contributions to the gravitons on H^4. Taking into account the $\mathcal{C}^s$ gradation, the averaged metric perturbation (5.83) is non-vanishing only for the modes $\mathcal{A}_a^{(3)}$ and $\mathcal{A}_a^{(2')}$.

Spin 1 graviton $\mathcal{A}_a^{(2)}$. Here, we observe

$$H_{ab}[\mathcal{A}^{(2)}] = -r^2R^2\{P_{ab},\phi^{(1)}\} - r^2R^2\left(\nabla_a\mathcal{A}_b^{(2)} + \nabla_b\mathcal{A}_a^{(2)}\right) + r^2\left(x_a\mathcal{A}_b^{(2)} + x_b\mathcal{A}_a^{(2)}\right)$$
$$= -r^2R^2\left(\nabla_a\mathcal{A}_b^{(2)} + \nabla_b\mathcal{A}_a^{(2)}\right) \tag{5.91}$$

such that the averaging yields

$$h_{ab}[\mathcal{A}^{(2)}] = \alpha_1\left(\nabla_a(\Box - 2r^2)\phi_b + \nabla_b(\Box - 2r^2)\phi_a\right)\,. \tag{5.92}$$

This has the form of pure gauge (diffeomorphism) contributions. Since the $\mathcal{A}^{(2)}$ modes are pure gauge, they are not physical in the present model.

Spin 1 graviton $\mathcal{A}_a^{(3)}$. Similarly, we have

$$H_{ab}[\mathcal{A}^{(3)}] = \theta^{ad}\theta^{bf}\left(\eth_f\phi_d + \eth_d\phi_f\right) \tag{5.93}$$

such that averaging yields

$$h_{ab}[\mathcal{A}^{(3)}] = -\alpha_1\left(\nabla_a\phi_b + \nabla_b\phi_a\right)\,. \tag{5.94}$$

Physical spin 1 gravitons. For the spin 1 eigenmodes $\mathcal{B}^{(i)}$ of (5.47), we therefore obtain the following physical gravitons:

$$\begin{aligned}\widetilde{h}_{ab}[\mathcal{B}^{(1)}] &= \widetilde{h}_{ab}[\mathcal{B}^{(4)}] = 0\,,\\ \widetilde{h}_{ab}[\mathcal{B}^{(2)}] &= \alpha_1(1-\alpha_1)\Big(\nabla_a(\Box - 2r^2)\phi_b + \nabla_b(\Box - 2r^2)\phi_a\Big)\,,\\ \widetilde{h}_{ab}[\mathcal{B}^{(3)}] &= -\alpha_1\left(\nabla_a\phi_b + \nabla_b\phi_a\right)\,.\end{aligned} \tag{5.95}$$

Hence there is indeed a physical spin 1 mode $\widetilde{h}_{ab}[\mathcal{B}^{(3)}]$ contributing to the metric fluctuations. Nevertheless, since it has the form of pure gauge (diffeomorphism) contributions, it will decouple from a conserved energy-momentum tensor $T_{\mu\nu}$.

5.6.3. Spin 2 gravitons

Finally, we consider the spin 2 fluctuations of the background and evaluate their associated graviton modes.

Spin 2 graviton $\mathcal{A}_a^{(1)}$. Since $\mathcal{A}_a^{(1)} = \eth_a\phi^{(2)}$ with $\phi^{(2)} = \{x^a, \{x^b, \phi_{ab}\}\} \in \mathcal{C}^2$, we have

$$\begin{aligned}H_{ab} &= \theta^{da}\{\eth_d\phi^{(2)}, x^b\} + (a \leftrightarrow b)\\ &= \{x^b, \{\theta^{ad}\eth_d\phi^{(2)}\} - \{\theta^{da}, x^b\}\eth_d\phi^{(2)} + (a \leftrightarrow b)\\ &= \{x^b, \{x^a, \phi^{(2)}\}\} - \{\theta^{da}, x^b\}\eth_d\phi^{(2)} + (a \leftrightarrow b)\,.\end{aligned} \tag{5.96}$$

The second term drops out in the projection to $\mathcal{C}^0$, and using (4.29) twice one finds

$$h_{ab}[\mathcal{A}^{(1)}] = \frac{4}{L_{NC}^4}[\{x^b, \{x^a, \phi^{(2)}\}\}]_0 = \frac{2}{R^2r^2}\hat{\alpha}_1\hat{\alpha}_2\phi_{ab}\,. \tag{5.97}$$

Spin 2 graviton $\mathcal{A}_a^{(2)}$. For $\mathcal{A}_a^{(2)} = \{x^c, \phi_{bc}\} \in \mathcal{C}^1 \oplus \mathcal{C}^3$, it follows that $H_{ab} \in \mathcal{C}^{\mathrm{odd}}$ and therefore

$$h_{ab} \propto [H_{ab}]_0 = 0\,. \tag{5.98}$$

In fact this is a pure gauge mode in the model.

Spin 2 graviton $\mathcal{A}_a^{(3)}$. Next, consider $\mathcal{A}_a^{(3)} = \{x^c, \phi_{ac}\} \in \mathcal{C}^1$. Then $H_{ab} \in \mathcal{C}^{\mathrm{odd}}$, and again

$$h_{ab} \propto [H_{ab}]_0 = 0\,. \tag{5.99}$$

Spin 2 graviton $\mathcal{A}_a^{(4)}$. Finally, consider the mode $\mathcal{A}_a^{(4)} = \theta^{ae}\{x^c, \phi_{ec}\} \in \mathcal{C}^2$. Then

$$\begin{aligned}H_{ab} &= \theta^{da}\{\theta^{de}\{x^c, \phi_{ec}\}, x^b\} + (a \leftrightarrow b)\\ &= \theta^{da}\theta^{de}\{\{x^c, \phi_{ec}\}, x^b\} + \theta^{da}\{\theta^{de}, x^b\}\{x^c, \phi_{ec}\} + (a \leftrightarrow b)\\ &= r^2R^2\{\{x^c, \phi_{ac}\}, x^b\} - r^2\theta^{ba}\{x^c, \phi_{ec}\}x^e + (a \leftrightarrow b)\\ &= -r^2R^2\{x^b, \{x^c, \phi_{ac}\}\} + (a \leftrightarrow b)\end{aligned} \tag{5.100}$$

using (4.15). Recall that (4.29) implies $[\{x^b, \{x^c, \phi_{ac}\}]_0 = -\hat{\alpha}_1\phi_{ab}$, and therefore

$$h_{ab}[\mathcal{A}^{(4)}] = 2\alpha_1(\Box - 2r^2)\phi_{ab}\,. \tag{5.101}$$

Physical gravitons. Computing the gravitons for the eigenmodes $\mathcal{B}^{(i)}$, we find

$$\begin{aligned}&\widetilde{h}_{ab}[\mathcal{B}^{(i)}]=0 \qquad \text{for} \quad i=1,2,3\\&\widetilde{h}_{ab}[\mathcal{B}^{(4)}]=2\alpha_1(\Box-2r^2)\phi_{ab}, \qquad \eth^a\widetilde{h}_{ab}[\mathcal{B}^{(4)}]=0=\nabla^a\widetilde{h}_{ab}[\mathcal{B}^{(4)}]\end{aligned} \tag{5.102}$$

using (D.26). The trivial result for $\mathcal{B}^{(i)}$, $i=2,3$, is obvious, as the individual contributions for $\mathcal{A}^{(i)}$, $i=2,3$, vanish. However, the vanishing contribution of $\mathcal{B}^{(1)}$ is the result of a non-trivial cancellation of the contributions from $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(4)}$.

In summary, the physical fields contributing to the metric fluctuations are a spin 2 field, a spin 1 field, and a spin 0 field. This is somewhat reminiscent of scalar-vector-tensor gravity. The spin 0 and spin 2 modes both satisfy the de Donder gauge.

To understand the present organization into spin modes, recall that the linearized metric fluctuations h_{ab} decompose in general as

$$h_{ab}=h^{(2)}_{ab}+\nabla_a\xi_b+\nabla_b\xi_a+\frac{1}{4}\eta_{ab}h \tag{5.103}$$

where $h^{(2)}_{ab}$ is a divergence-free, traceless spin 2 tensor. This corresponds to our spin 2, spin 1 and spin 0 contribution to the graviton; note that ξ_a contains another spin 0 (divergence) mode. While the ξ_a fields are unphysical pure gauge modes, the spin 0 part h is a physical field which is in general sourced by the trace of the energy-momentum tensor. In the Einstein-Hilbert action, this spin 0 field enters with the “wrong” sign, cf. [35]. This does not happen here, which is certainly welcome for the quantization of the model.

5.7. Classical action for metric fluctuations

Having defined the notion of physical graviton in (5.85), an effective 4-dimensional action for $\widetilde{h}^{ab}$ is desirable. By writing the trace as an integral as in (3.77), one can express the (gauge-fixed) kinetic term for $\mathcal{B}^{(4)}$ in terms of $\widetilde{h}_{ab}\equiv\widetilde{h}_{ab}[\mathcal{B}^{(4)}]$ as follows:

$$\begin{aligned}S_2&=\frac{1}{g^2}\int\rho\,\mathcal{B}^{(4)}\mathcal{D}^2\mathcal{B}^{(4)}=\frac{1}{g^2}\alpha_1r^2R^2\int\rho\,\phi^{(2)}_{ab}[\mathcal{B}^{(4)}](\Box-2r^2)^2\phi^{(2)}_{ab}[\mathcal{B}^{(4)}]\\&=\frac{1}{4\alpha_1g^2_{\mathrm{YM}}L^4_{NC}}\int\widetilde{h}_{ab}[\mathcal{B}^{(4)}]\widetilde{h}_{ab}[\mathcal{B}^{(4)}]\end{aligned} \tag{5.104}$$

where g_{YM} is the dimensionless Yang–Mills/Maxwell coupling constant (5.78). Superficially, this looks like a mass term for the graviton; however this is only the spin two mode, which is by definition invariant under diffeomorphisms. Hence (5.104) could also be viewed as the quadratic contribution to the cosmological constant in GR.[17]

Taking into account a coupling to matter of the form $\delta_hS=\frac{1}{2}\int\widetilde{h}_{ab}T^{ab}$, the equations of motion for $\widetilde{h}_{ab}$ become

$$\widetilde{h}_{ab}[\mathcal{A}^{(4)}]=-\frac{4}{3}\frac{g^2}{\rho L^4_{NC}}T_{ab}=-\frac{1}{3}g^2_{\mathrm{YM}}L^4_{NC}T_{ab}\,. \tag{5.105}$$

Clearly $\widetilde{h}_{ab}$ is not propagating, but acts like an auxiliary field which tracks T_{ab}. As a consequence, the pure matrix model action (5.1) does not lead to gravity on H^4, similar to the case of S^4_N

[17] Hence a large positive mass would not imply large curvature but rather a short range of these modes. See e.g. [36] for a related discussion.

[2]. Nevertheless, the action (5.1) does define a non-trivial, and apparently not pathological, spin 2 theory in 4 dimensions with a propagating spin 2 field ϕ_{ab}, which should be suitable for quantization. Gravity may then arise upon quantization, as discussed next.

5.8. *Induced gravity*

At first sight it may seem disappointing that gravity does not arise from the classical action. On the other hand, since classical GR is not renormalizable, it should presumably be viewed as a low-energy effective theory. Adopting this point of view, it is reasonable that the starting point of an underlying quantum theory can be very different at the classical level, as for instance in the approach advocated here. This train of though is exactly the idea of *emergent gravity*.[18]

As soon as quantum effects in the matrix model are taken into account, the effective metric $\widetilde{h}_{ab}$ will unavoidably acquire a kinetic term, and therefore propagate. More specifically, it is well-known that induced gravity terms arise at one loop, upon integrating out fields that couple to the effective metric [17,18,38]. The induced terms include the cosmological constant and Einstein-Hilbert terms. The maximal supersymmetry of the underlying model[19] along with the finite density of states of the solution strongly suggests that the model is UV finite and "almost-local". Moreover, the usual large contribution to the cosmological constant $\int\sqrt{g}\Lambda^4$ is avoided here, cf. the one-loop computation in [3]. Canceling also the induced Einstein-Hilbert term is more subtle,[20] and it is plausible that the supersymmetry breaking H^4 background does lead to an induced Einstein-Hilbert term with scale $\tilde{\Lambda} = O\left(\frac{1}{r}\right)$.

Motivated by these considerations, one may add a term $\int \sigma\tilde{\Lambda}^2\widetilde{h}_{ab}\eth\cdot\eth\widetilde{h}^{ab}$ to the action (5.104), with $\sigma = \pm 1$, such that the total action coupled to matter reads

$$S = \int \sigma\tilde{\Lambda}^2\widetilde{h}_{ab}\eth\cdot\eth\widetilde{h}^{ab} + \frac{4}{3g_{\rm YM}^2 L_{NC}^4}\int \widetilde{h}_{ab}\widetilde{h}_{ab} + \frac{1}{2}\int \widetilde{h}_{ab}T^{ab} \,. \tag{5.106}$$

The equation of motion for $\widetilde{h}_{ab}$ are then

$$\left(\eth\cdot\eth + \frac{4}{3\sigma g_{\rm YM}^2 L_{NC}^4\tilde{\Lambda}^2}\right)\widetilde{h}_{ab} = -\frac{1}{4\sigma\tilde{\Lambda}^2}T_{ab} \tag{5.107}$$

where $\tilde{\Lambda}$ is the effective cutoff scale set by induced gravity. For $\sigma = -1$, this is indeed a reasonable equation for linearized gravity, with the effective Newton constant

$$8\pi G_N = \frac{1}{8\tilde{\Lambda}^2} \tag{5.108}$$

and mass scale

[18] In fact, it is known that the Type IIB bulk gravity in the IKKT model arises only at one loop [37]. However, this is a different issue, since the present degrees of freedom are only 4-dimensional.

[19] This really requires the maximal supersymmetry of the IKKT model, otherwise UV/IR mixing effects will render the model strongly non-local and probably pathological, cf. [39,40].

[20] On Moyal–Weyl backgrounds, $\mathcal{N} = 1$ SUSY is sufficient to cancel the induced "would-be" cosmological constant term, while the induced Einstein–Hilbert term is only canceled in the $\mathcal{N} = 4$ case [41,42]. This is reflected by the absence of UV/IR mixing. Here the background and the explicit mass term induce a spontaneous and soft breaking of $\mathcal{N} = 4$ SUSY. Nevertheless, the suggested scenario seems reasonable.

$$m^2 = O\left(\frac{1}{g_{\rm YM}^2 L_{NC}^4 \tilde{\Lambda}^2}\right). \tag{5.109}$$

The mass scale can become very small $m^2 = O\left(\frac{1}{R^2}\right)$ if $\tilde{\Lambda} = O\left(\frac{1}{r}\right)$ and n is large, or upon projection to the Minkowski space-time $\mathcal{M}^{3,1}$, where the universe grows in time. Of course, the mass term will acquire quantum corrections too, which will be suppressed by supersymmetry. It would be desirable to study this in more detail elsewhere.

Even though such a mass term might be interpreted in terms of a cosmological constant in linearized GR, its meaning here is somewhat different. As in GR, a proper interpretation requires the full non-linear theory. However, it is plausible that a positive mass term may simply imply an IR cutoff for gravity here, while the large-scale structure of the background solution might not be affected. Therefore a small, but non-zero mass term is quite welcome in the presented setting to ensure stability, while the large-scale cosmology would be determined by the background solution, as illustrated in section 6.

5.9. *Local gauge transformations*

Among the higher-spin gauge transformations $\delta_\Lambda(x^a + \mathcal{A}^a) := \{x^a + \mathcal{A}^a, \Lambda\}$ generated by $\Lambda \in \mathcal{C}$, consider the spin 1 gauge transformations generated by

$$\Lambda^{(1)} = \{x^a, v_a\} \; = \; \theta^{ab}\eth_b v_a \quad \in \mathcal{C}^1 \tag{5.110}$$

with $v_a(x)$ a divergence-free vector field. These correspond to (volume-preserving) diffeomorphisms on H^4. The action on scalar functions $\phi(x)$ reads

$$\delta_\Lambda \phi = \{\phi(x), \Lambda^{(1)}\} = \{\phi(x), \theta^{ab}\eth_b v_a\} \tag{5.111}$$

so that the action on vector fluctuations is

$$\delta_\Lambda \mathcal{A}_a = \delta_\Lambda x_a + \{\mathcal{A}_a, \Lambda^{(1)}\} \tag{5.112}$$

with[21]

$$\begin{aligned}\delta_\Lambda x_a &= \{x_a, \Lambda^{(1)}\} = \{x_a, \Lambda\}_0 + \{x_a, \Lambda^{(1)}\}_2 \\ &= \alpha_1(\Box - 2r^2)v_a + \mathcal{A}_a^{(2')}[\Lambda^{(1)}]\end{aligned} \tag{5.113}$$

using (4.9). The first term describes a diffeomorphism corresponding to the vector field $\tilde{v}_a = \alpha_1(\Box - 2r^2)v_a$. The second term accounts for the spin 1 pure gauge mode $\mathcal{A}_a^{(2')}$ as discussed in section 5.1, whose contribution to the graviton $\widetilde{h}_{ab}$ was computed in (5.91). The higher-spin gauge transformations could be worked out similarly.

Since there is only one such gauge invariance, but several fields for each spin, one may worry about the consistency of the model. However, recall that the gauge-fixed action (5.71) has been proven to be well-defined and non-degenerate in section 5.4. Hence there is no problem at least in the Euclidean setting. This is due to the special origin of the fluctuation modes in $\mathrm{End}(\mathcal{H}_n)$, see (3.55).

[21] Note that $\{\mathcal{A}_a, \Lambda^{(1)}\}$ is not necessarily tangential. However that term vanishes in the semi-classical limit, and is significant only for nonabelian gauge fields which we do not consider. The proper treatment is of course to impose the non-linear constraint $Y_a Y^a = -R^2$, which would restore gauge invariance.

6. Lorentzian quantum space-times from fuzzy H_n^4

Having disentangled the fluctuations on H_n^4, we would like to apply these tools to the more interesting cosmological space-time solutions $\mathcal{M}^{3,1}$. Since the latter is obtained by a projection considered in section 3.2, many considerations remain valid. Most importantly, the fluctuation modes originate from the *same* $\mathrm{End}(\mathcal{H}_n)$ such that we can rely on the same spin operator $\mathcal{S}^2$, and our classification can be carried over. Moreover, the tangential fluctuations on H_n^4 are in one-to-one correspondence to the full set of fluctuation modes on $\mathcal{M}^{3,1}$, as will be shown below. The symmetry group is reduced to $SO(3,1)$ instead of $SO(4,1)$, which is weaker, but should still be very useful.

6.1. Cosmological space-time solutions

By projecting fuzzy H_n^4 onto the 0123 plane via Π of (2.30) i.e. by keeping the $Y^\mu = \mathcal{M}^{\mu a}\alpha_a$ for $\mu = 0,1,2,3$ and dropping Y^4, we obtain $(3+1)$-dimensional fuzzy space-time solutions. Since the embedding metric $\eta^{\mu\nu}$ is compatible with $SO(3,1)$, we have

$$\begin{aligned}[][Y_\rho,[Y^\rho,Y^\mu]] &= i(\alpha\cdot\alpha)[Y_\rho,\mathcal{M}^{\rho\mu}] = -i(\alpha\cdot\alpha)[\mathcal{M}^{\rho\mu},Y_\rho] \\ &= (\alpha\cdot\alpha)\begin{cases} Y^\mu, & \mu\neq\rho \\ 0, & \mu=\rho \end{cases} \qquad \text{(no sum)} \end{aligned} \tag{6.1}$$

such that

$$\Box_Y Y^\mu = [Y^\rho,[Y_\rho,Y^\mu]] = 3(\alpha\cdot\alpha)\,Y^\mu\,. \tag{6.2}$$

Depending on $\alpha\cdot\alpha$ we obtain three different types of quantized space-time solutions with Minkowski signature in the IKKT model with mass term. These are:

$$\begin{aligned} \Box_X X^\mu &= -3r^2\,X^\mu\,, \\ \Box_T T^\mu &= \frac{3}{R^2}\,T^\mu\,, \\ \Box_Z Z^\mu &= 0\,. \end{aligned} \tag{6.3}$$

Choosing a positive mass term to ensure stability, we focus on the solution

$$Y^\mu = X^\mu, \qquad r^2 = \frac{1}{3}m^2\,. \tag{6.4}$$

This is the homogeneous and isotropic quantized FLRW cosmological space-time $\mathcal{M}_n^{3,1}$ with $k=-1$ introduced[22] in [1]. Here m^2 sets the scale r^2, while n remains undetermined. These backgrounds are $SO(3,1)$-covariant, which is the symmetry respected by $\Box_X$.

6.2. Semi-classical geometry

We first recall the semi-classical limit of this space [1], with x^μ for $\mu = 0,1,2,3$ as coordinates on $\mathcal{M}$. By $SO(3,1)$-invariance, we can always consider the local reference point ξ on H^4 resp. $\mathcal{M}$

[22] We change notation from [1], where Y^1 was dropped instead of Y^4.

$$\xi = (x^0, 0, 0, 0, x^4) \overset{\Pi}{\to} (x^0, 0, 0, 0), \qquad x^0 = R\cosh(\eta), \;\; x^4 = R\sinh(\eta)\,. \tag{6.5}$$

Globally, we have the following constraints

$$\begin{aligned} x_\mu x^\mu &= -R^2 - x_4^2 = -R^2\cosh^2(\eta)\,, \\ t_\mu t^\mu &= r^{-2}\cosh^2(\eta)\,, \\ t_\mu x^\mu &= 0, \qquad \mu, \nu = 0, \dots, 3 \end{aligned} \tag{6.6}$$

where η will be a global "cosmic" time coordinate. From the radial constraint $x_a x^a = -R^2$ on H^4 one deduces $\{x_a x^a, x^\mu\} = 0$, which further implies

$$0 = x_a m^{a\mu} = x_\nu m^{\nu\mu} + x_4 m^{4\mu}\,. \tag{6.7}$$

This establishes a relation between the momenta and the t^μ,

$$t^\mu = \frac{1}{R} m^{\mu 4} = \frac{1}{Rr^2 x^4} x_\nu \theta^{\nu\mu} \overset{\xi}{=} \frac{1}{Rr^2} \frac{1}{\tanh(\eta)} \theta^{0\mu}\,. \tag{6.8}$$

Furthermore, the self-duality constraint (3.58c) reduces to[23]

$$\begin{aligned} t^i &= \frac{1}{R} m_{i4} = \frac{1}{nRr^3} \epsilon_{abci4} \theta^{ab} x^c \overset{\xi}{=} \frac{1}{nr^3} \cosh(\eta) \epsilon^{ijk} \theta^{jk}, \\ t^0 &\overset{\xi}{=} 0\,, \end{aligned} \tag{6.9}$$

where the last equation is simply a consequence of $x_\mu t^\mu = 0$. Therefore t^μ describes a space-like S^2 with radius $r^{-2}\cosh^2(\eta)$. Conversely, the above relations allow to express $\theta^{\mu\nu}$ in terms of the momenta t^μ as follows

$$\begin{aligned} \theta^{ij} &= \frac{nr^3}{2\cosh(\eta)} \varepsilon^{ijk} t^k\,, \\ \theta^{0i} &= Rr^2 \tanh(\eta)\, t^i\,. \end{aligned} \tag{6.10}$$

By means of $R \sim \frac{1}{2} nr$, one can summarize (6.10) neatly:

$$\theta^{\mu\nu} = r^2 R\, \eta_\alpha^{\mu\nu}(x)\, t^\alpha\,, \tag{6.11}$$

where $\eta_\alpha^{\mu\nu}(x)$ is a $SO(3,1)$-invariant tensor field on $\mathcal{M}^{3,1}$, which is analogs of the t'Hooft symbols. Note that $\theta^{0i} \gg \theta^{ij}$ for late times $\eta \gg 1$; this reflects the embedding of $H^4 \subset \mathbb{R}^{4,1}$ which approaches the light cone at late times. Thus space is almost commutative, but space-time is not. Nevertheless the effects of non-commutativity will be weakened due to the averaging on S^2. Finally the constraint (3.58d) reads

$$\gamma^{\alpha\beta} := \eta_{\mu\nu} \theta^{\mu\alpha} \theta^{\nu\beta} = \frac{L_{NC}^4}{4} (\eta^{\alpha\beta} + \frac{1}{R^2} x^\alpha x^\beta - R^2 t^\alpha t^\beta) \tag{6.12}$$

which at the chosen reference point yields

[23] Note that this form only applies in the special $\mathfrak{so}(3,1)$ adapted frame, and it is not generally covariant; of course on Minkowski manifolds, there is no notion of self-duality. However there can be a $SO(3,1)$-invariant relation as above which holds in the preferred cosmological frames, and this is what happens here. This is one reason why it is important to *not* have full Poincare covariance in the Minkowski case.

$$
\begin{aligned}
\gamma^{ij} &= \frac{L_{NC}^4}{4}\,(\delta^{ij} - R^2\, t^i t^j)\,, \\
\gamma^{00} &= \frac{L_{NC}^4}{4}\,\sinh^2(\eta)\,, \\
\gamma^{0j} &= 0\,.
\end{aligned}
\tag{6.13}
$$

Averaging and effective metric on $\mathcal{M}^{3,1}$. An effective metric for scalar fields $\phi(x)$ on $\mathcal{M}^{3,1}$ can be defined by the quadratic action (5.79). Looking at (6.12), we note that $\gamma^{\alpha\beta}$ contains the term $t^\alpha t^\beta$, which is not constant on the fiber S^2. By averaging over the fiber, one obtains the following result [1]

$$
\begin{aligned}
[\gamma^{ij}]_0 &= \frac{L_{NC}^4}{4}\,\delta^{ij} - [t^i t^j]_0 = \frac{L_{NC}^4}{12}\,\big(3 - \cosh^2(\eta)\big)\delta^{ij}\,, \\
[\gamma^{00}]_0 &= \frac{L_{NC}^4}{4}\,\sinh^2(\eta), \qquad [\gamma^{0i}]_0 = 0\,.
\end{aligned}
\tag{6.14}
$$

Note the signature change at $\cosh^2(\eta) = 3$ which marks the Big-Bang in this model, and the large pre-factors which grow in time η. Taking into account the conformal factor in the effective metric $G^{\mu\nu}$ (5.81), one obtains the cosmic scale parameter $a(t) \propto t$ for late times, corresponding to a coasting universe [1].

However we have not yet shown that this metric $G^{\mu\nu}$ governs all of the low-energy physics, and that there are no tachyonic or ghost modes. The large local symmetry of the model and the universal structure of the Yang–Mills action should help to elaborate the full dynamics. Here we only take some steps in that direction: we establish a precise correspondence between the fluctuation modes as well as a close relation between the action of both spaces.

6.3. *Wave-functions, higher-spin modes and constraints*

In this section we briefly comment on the fluctuation modes on $\mathcal{M}^{3,1}$. The space of functions $\mathrm{End}(\mathcal{H}_n)$ on $\mathcal{M}^{3,1}$ is the same as on H_n^4, meaning that the decomposition (3.55) remains valid and is truncated at order n. The modes will still be considered as functions (or sections of higher-spin bundles) on H^4, such that a representation as in (3.42) is expected to hold. Consequently, the modes can be interpreted as functions (or higher-spin modes) on[24] $\mathcal{M}^{3,1}$ via Π of (2.30). The $\phi_{ab}(x)$ etc. then define some higher-rank field on $\mathcal{M}^{3,1}$. In the following, we will only address a few basic points.

6.4. *Tangential fluctuation modes, relation with H^4 and $SO(4,1)$*

Now consider fluctuations $y^\mu = x^\mu + \mathcal{A}^\mu$ around $\mathcal{M}^{3,1}$. The first observation is that these four fluctuation modes $\mathcal{A}^\mu$, $\mu = 0, \ldots, 3$ are in one-to-one correspondence with the tangential fluctuations on H^4. To see this, recall that tangential fluctuations on H^4 satisfy by definition[25] the constraint

[24] We will ignore the dependence on two sheets $\mathcal{M}^\pm$ for simplicity.

[25] A gauge-invariant constraint would be $(X^a + \mathcal{A}^a)(X_a + \mathcal{A}_a) = -R^2$. For the present purpose, its linearized form is what we want.

$$\mathcal{A}_a x^a = 0, \qquad \mathcal{A}_4 = -\frac{x_\mu}{x_4}\mathcal{A}^\mu \,, \tag{6.15}$$

with $\mathcal{A}_a \in \mathrm{End}(\mathcal{H})$. To associate a general fluctuation mode on $\mathcal{M}^{4,1}$ one simply drops $\mathcal{A}^4$, and conversely $\mathcal{A}^4$ can be recovered from $\mathcal{A}^\mu$ via (6.15). Hence there is a correspondence of tangential fluctuations

$$\begin{array}{ccc} \mathcal{A}^a \text{ on } H^4 & \longleftrightarrow & \mathcal{A}^\mu \text{ on } \mathcal{M}^{3,1} \\ {\scriptstyle SO(4,1)}\big\downarrow & & \\ \mathcal{A}^a \text{ on } H^4 & \longleftrightarrow & \mathcal{A}^\mu \text{ on } \mathcal{M}^{3,1} \end{array} \tag{6.16}$$

Since the maps are invertible, an $SO(4,1)$-action is defined on the fluctuations $\mathcal{A}^\mu$ on $\mathcal{M}^{3,1}$, which, however, is not an isometry and not unitary. Nevertheless it acts as a structural group, and organization developed for H^4 in the previous sections remains applicable. As a consequence, configurations in the $\mathcal{M}^{3,1}$ model can be mapped one-to-one to configurations in the H^4 model. Similarly, higher-rank tangential tensors on H^4 such as the gravitons

$$h_{ab}x^a = 0 \tag{6.17}$$

can be mapped one-to-one to tensors $h_{\mu\nu}$ on $\mathcal{M}^{3,1}$, and the missing components h_{ab} are uniquely determined from the $h_{\mu\nu}$. In the same vein, all internal fluctuations on S^2 will be organized in a $SO(4,1)$-covariant way as on H^4. This relation is somewhat analogous to a Wick rotation.

Action and dynamics. The matrix model provides again an action for the fluctuation modes $\mathcal{A}^\mu$, which has the same structure as in section 5.4,

$$S_{\mathcal{M}}[\mathcal{A}] = \int \mathcal{A}_\mu \left(\Box_{\mathcal{M}} - 2\mathcal{I} + \frac{1}{2}\mu^2\right)\mathcal{A}^\mu \tag{6.18}$$

upon gauge fixing. The matrix Laplacian on $\mathcal{M}^{3,1}$ is related to the one on H^4 through

$$\Box_{\mathcal{M}} = \Box_H - [X^4,[X_4,\cdot]] = [X^\mu,[X_\mu,\cdot]] \sim -\frac{L_{NC}^4}{4}\,\gamma^{\mu\nu}\partial_\mu\partial_\nu + \dots \,. \tag{6.19}$$

We can utilize the same mode expansion in terms of $\mathcal{A}_\mu^{(i)}$ as in section 5.1,

$$\begin{aligned} \mathcal{A}_\mu^{(1)} &= \eth_\mu \phi^{(s)} \ \in \mathcal{C}^s, \qquad \phi^{(s)} = \{x^a, \phi_a^{(s)}\} = \{x^\mu, \phi_\mu^{(s)}\} + \{x^4, \phi_4^{(s)}\} \ \in \mathcal{C}^s \\ \mathcal{A}_\mu^{(2)} &= \theta^{\mu b}\eth_b \phi^{(s)} \ = \{x^\mu, \phi^{(s)}\} \quad \in \mathcal{C}^{s+1}\oplus\mathcal{C}^{s-1} \\ \mathcal{A}_\mu^{(3)} &= \phi_\mu^{(s)} \ \in \mathcal{C}^{s-1}, \\ \mathcal{A}_\mu^{(4)} &= \theta^{\mu b}\phi_b^{(s)} \ \in \mathcal{C}^s \,, \end{aligned} \tag{6.20}$$

which is $SO(3,1)$-covariant. As explained in section 5.2, the irreducibility constraints, i.e. transversality and tracelessness, can be implemented as appropriate for $\mathcal{M}^{3,1}$ without changing the setup. The relation (4.9) still applies; for example

$$\{x_\mu, \phi^{(1)}\}_0 = -\frac{2}{3}\phi_\mu + \frac{R^2}{3}\eth^c\eth_c\phi_\mu \,. \tag{6.21}$$

Note that $\eth^c\eth_c$ is the *Euclidean* Laplace operator on H^4, even though we are working in the Minkowski case. Hence the right-hand side of (6.21) amounts to some field redefinition. In the same vein, the higher-derivative terms in the action (5.72) for the rank s tensor fields $\phi_{a_1\ldots a_2}$

amount to field redefinitions. Therefore one should expect that these higher-derivative terms do *not* lead to new degrees of freedom or ghosts.

On the other hand it might be tempting to use a $SO(3,1)$-covariant formalism, where e.g. $\{x^a, \phi_a^{(s)}\}$ in (6.20) is replaced by $\{x^\mu, \phi_\mu^{(s)}\}$. However then some identities are lost, and it remains to be seen which formalism is more advantageous.

7. Conclusion and outlook

In this article we provide a careful and detailed analysis of the fluctuation modes on fuzzy H_n^4 as a background in Yang–Mills matrix models, focusing mainly on the semi-classical case. While the results are largely analogous to the case of S_N^4 [2], the present approach based on a suitable Poisson calculus is more transparent and fairly close to a standard field-theory treatment. The intrinsic structure of these quantum spaces is responsible for obtaining a higher-spin gauge theory, which is fully $SO(4,1)$-covariant. The key feature is the equivariant bundle structure, which leads to a transmutation of would-be Kaluza–Klein modes into higher-spin modes.

Summary. Let us summarize the main points: A suitable set of representations for the construction of H_n^4 is identified as the *minireps* or *doubleton* $\mathcal{H}_n$, for which we recall the oscillator realization in section 3. The first major step is a classification of the fuzzy algebra of functions $\mathrm{End}(\mathcal{H}_n)$, which relies on two pillars: (i) the construction of a spin Casimir invariant $\mathcal{S}^2$ which measures the intrinsic angular momentum on the S_n^2 fiber, and (ii) the statement that the quantization map (3.32) is surjective. This provides the basis for the expansion (3.42) of a generic function in $\mathrm{End}(\mathcal{H}_n)$, which is a finite expansion in the generators associated to the fiber. More precisely, $\mathrm{End}(\mathcal{H}_n)$ decomposes into a *finite* set of higher-spin sectors $\mathcal{C}^s$ (3.55), labeled by the spin Casimir. In the semi-classical limit, these become modules over the algebra of functions on H^4, which are identified with tangential, divergence-tree, traceless rank s tensor fields on H^4.

The second major step is the development of a suitable differential calculus built upon derivations $\eth_a$ defined via the Poisson bracket in (3.64). This provides the tools to work explicitly with the generic spin s modes on a non-compact space.

Having in mind the IKKT matrix model, we observe that H_n^4 is a solution of Yang–Mills matrix models with mass term, and classify the fluctuation modes around an H_n^4 background in section 5. Building on the understanding of $\mathrm{End}(\mathcal{H}_n)$, we find four tangential (5.29) and one radial fluctuation modes for each spin $s \geq 1$. We find the explicit eigenmodes $\mathcal{B}^{(i)}$ of the differential operator $\mathcal{D}^2$, see (5.65), which governs the fluctuations in the matrix model. It turns out that the tangential modes are stable, due to positivity results on their kinetic terms.

Next, we identify the physical graviton (5.85) as linearized fluctuation of the effective metric (5.81) around the H_n^4 background, and we compute the associated graviton modes for spin $s = 0, 1, 2$. The gravitons at spin $s = 0, 2$ naturally satisfy the de Donder gauge. However, it turns out that the spin 2 graviton behaves as an auxiliary field, at least at the classical level. A more interesting gravitational behavior should be obtained by including quantum corrections, leading to induced gravity terms. We briefly discuss this scenario in section 5.8.

Considering H_n^4 as a starting point towards the fuzzy space-time $\mathcal{M}_n^{3,1}$, these issues are however less important. Since $\mathcal{M}_n^{3,1}$ is obtained from H_n^4 by a projection, the fuzzy algebra of functions for $\mathcal{M}_n^{3,1}$ coincides with $\mathrm{End}(\mathcal{H}_n)$, and our results provide a useful set of tools. As first steps, we briefly discuss the geometry and the organization of higher-spin modes of $\mathcal{M}_n^{3,1}$, and establish a relation between tangential fluctuation on H_n^4 and $\mathcal{M}_n^{3,1}$ in section 6.

Discussion and outlook. From a physics point of view, the results may seem a bit disappointing in the sense that the spin 2 modes do not lead to a propagating graviton at the classical level. However gravity could be restored in the quantum case, where induced gravity terms arise. The most encouraging result is that the tangential fluctuations are stable and do not lead to ghost-like modes. This is an improvement over GR where the off-shell conformal modes have the wrong sign, and arguably also over quadratic gravity where ghost-like modes arise at least superficially, cf. [33,34]. On the other hand, the radial modes are unstable here, which however could be cured by a radial constraint.

There are several issues which deserve to be studied further. For example, the Poisson calculus developed here should be extended to the fully non-commutative case. Likewise, the relation of the present higher-spin gauge theory with Vasiliev theory should be clarified. In view of the H^4 geometry, it is natural to contemplate possible applications of holography. Some of the structural statements in sections 3.5 and 3.6 would deserve a more rigorous treatment. Furthermore, the 1-loop computation in [3] could easily be adapted, since H^4 is locally very similar to S^4. This would allow to make more specific statements about the induced gravity terms, although to obtain the Einstein-Hilbert term may require a more refined approach. Finally, the minimal case $n = 0$ is very remarkable and special, because it does not correspond to a quantized symplectic space.

The main physics motivation for the present work is the close relation to the cosmological FLRW-type solutions $\mathcal{M}^{3,1}$ of [1], which are obtained from a projection of H_n^4. The fluctuation analysis on $\mathcal{M}^{3,1}$ can largely proceed along the same lines, with some important differences. In particular, the radial modes will disappear while the signature becomes Lorentzian. Furthermore, field redefinitions such as (5.101), which are responsible for the non-propagating nature of the graviton on H^4, should no longer cancel the propagator. Therefore $\mathcal{M}_n^{3,1}$ is a very promising candidate for a quantum space-time with interesting gravitational physics in the framework of matrix models. However, we postpone a detailed analysis of $\mathcal{M}^{3,1}$ to future work.

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Appendix A. Some aspects of $SO(4,2)$

The Lie algebra $\mathfrak{so}(4,2)$ is defined by

$$[M_{ab}, M_{cd}] = \mathrm{i}\,(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}) \ , \tag{A.1}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1,1,-1)$ with $a, b, \ldots = 0,1,2,3,4,5$. Unitary representations of $SO(4,2)$ are given by Hermitian $\mathcal{M}^{ab}$. The maximal compact subgroup of $SO(4,2)$ is $SU(2)_L \times SU(2)_R \times U(1)_E$, generated by the following generators:

$$L_m = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{mnl}M_{nl} + M_{m4}\right) \quad \longrightarrow \quad SU(2)_L$$

$$R_m = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{mnl}M_{nl} - M_{m4}\right) \quad \longrightarrow \quad SU(2)_R$$

with $m, n, l = 1, 2, 3$. They satisfy

$$\begin{aligned} [L_m, L_n] &= \mathrm{i}\varepsilon_{mnl} L_l \,, \\ [R_m, R_n] &= \mathrm{i}\varepsilon_{mnl} R_l \,, \\ [L_m, R_n] &= [E, L_n] = [E, R_n] = 0 \,. \end{aligned} \tag{A.2}$$

The $U(1)_E$ generator $E = M_{05}$ is the conformal Hamiltonian, whose spectrum is positive in a positive energy representation. Denoting the maximal compact Lie sub-algebra of $SU(2)_L \times SU(2)_R \times U(1)_E$ as $\mathcal{L}^0$, the conformal algebra $\mathfrak{g}$ has a three-graded decomposition

$$\mathfrak{g} = \mathcal{L}^+ \oplus \mathcal{L}^0 \oplus \mathcal{L}^-, \tag{A.3}$$

with respect to E, such that

$$[\mathcal{L}^0, \mathcal{L}^\pm] = \mathcal{L}^\pm, \qquad [E, \mathcal{L}^\pm] = \pm\mathcal{L}^\pm \,, \tag{A.4}$$

and $\mathcal{L}^\pm$ are the non-compact generators. The six roots of $\mathfrak{so}(6)_\mathbb{C}$ decompose accordingly into two compact roots $X^\pm_{\beta_i}$ and four non-compact roots $X_{\pm\hat{\alpha}_{ij}}$. The latter transform as $(2)_L \otimes (2)_R$ i.e. as complex vectors of $SO(4)$, and satisfy $(X_{\pm\hat{\alpha}_{ij}})^\dagger = -X_{\mp\hat{\alpha}_{ij}}$.

Spinorial representations of $SO(4,2)$ are obtained in terms of the $SO(3,1)$ gamma matrices γ_μ satisfying $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$ for $\mu, \nu = 0, 1, 2, 3$ and $\gamma_4 := \gamma_0\gamma_1\gamma_2\gamma_3$ as follows[26]:

$$\Sigma_{\mu\nu} := \frac{1}{4\mathrm{i}}\left[\gamma_\mu, \gamma_\nu\right] \qquad \Sigma_{\mu 4} := -\frac{\mathrm{i}}{2}\gamma_\mu\gamma_4 \qquad \Sigma_{\mu 5} := -\frac{1}{2}\gamma_\mu \qquad \Sigma_{45} := -\frac{1}{2}\gamma_4 \,. \tag{A.5}$$

We adopt the gamma matrix conventions

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \gamma_m = \begin{pmatrix} 0 & -\sigma_m \\ \sigma_m & 0 \end{pmatrix} \qquad \Rightarrow \qquad \gamma_4 = \mathrm{i}\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \tag{A.6}$$

where σ_m, $m = 1, 2, 3$ are the usual Pauli matrices. They satisfy

$$\begin{aligned} \gamma_a^\dagger &= -\gamma_b\eta^{ba} = \gamma_0\gamma_a\gamma_0^{-1}, \qquad a = 0, 1, 2, 3, 4 \\ \Sigma_{ab}^\dagger &= \Sigma_{a'b'}\eta^{aa'}\eta^{bb'} = \gamma_0\Sigma_{ab}\gamma_0^{-1}, \qquad a, b = 0, 1, 2, 3, 4, 5 \end{aligned} \tag{A.7}$$

as it should be. The universal covering group of $SO(4,2)$ is $SU(2,2)$, which is the group of 4×4 complex matrices with

$$U^{-1} = \gamma_0 U^\dagger \gamma_0^{-1} \tag{A.8}$$

which respects the indefinite sesquilinear form

$$\bar{\psi}_1\psi_2 = \psi_1^\dagger\gamma^0\psi_2 \,. \tag{A.9}$$

The 15-dimensional Lie algebra $\mathfrak{su}(2,2) = \mathfrak{so}(4,2)$ can thus be identified with the space of traceless complex 4×4 matrices Z^α_β with real structure

$$Z^\dagger = \gamma_0 Z \gamma_0^{-1} \,. \tag{A.10}$$

[26] To maintain a consistent notation for $SO(4,2)$, our γ^4 is what is usually called γ^5; this will not arise explicitly and should not cause confusion.

Appendix B. Conventions and identities for Gamma matrices

Using the sign conventions $\eta_{ab} = \mathrm{diag}(-1,1,1,1,1)$ and $\eta_{ab} = \mathrm{diag}(-1,1,1,1,1,-1)$, the Gamma matrices of $\mathfrak{so}(4,1)$ are

$$\{\gamma_a, \gamma_b\} = -2\eta_{ab}, \qquad a,b = 0,\dots,4 \tag{B.1}$$

such that $\gamma_0^2 = \mathbb{1}$ and $\gamma_0^\dagger = \gamma_0$, and more generally

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0^{-1} = -\eta_{ab}\gamma_b =: -\gamma^a. \tag{B.2}$$

Then

$$\gamma_a\gamma^a = -5\mathbb{1} \tag{B.3}$$

We can evaluate the $SO(4,1)$ intertwiner

$$\sum_{a,b\le 4} \Sigma_{ab}\otimes\Sigma^{ab} = \mathcal{C}^2[\mathfrak{so}(4,1)]_{(4)\otimes(4)} - 2\mathcal{C}^2[\mathfrak{so}(4,1)]_{(4)} \tag{B.4}$$

acting on

$$(4)\otimes(4) = \big((10)_S \oplus (6)_{AS}\big)_{\mathfrak{so}(4,2)} = \big((10)_S \oplus (5)_{AS} \oplus (1)_{AS}\big)_{\mathfrak{so}(4,1)} \tag{B.5}$$

Using the well-known eigenvalues of the quadratic Casimirs (which coincide with those of the compact group), it follows that

$$\Big(\sum_{a,b\le 4} \Sigma_{ab}\otimes\Sigma^{ab}\Big)_S = \mathbb{1}\,, \tag{B.6}$$

$$\Big(\sum_{a,b\le 5} \Sigma_{ab}\otimes\Sigma^{ab}\Big)_S = \frac{3}{2}\mathbb{1}\,. \tag{B.7}$$

This implies

$$\sum_{a\le 5}\big(\gamma_a\otimes\gamma^a\big)_S = -\mathbb{1} \qquad \text{and} \qquad \sum_{a,b\le 5}\Sigma_{ab}\Sigma^{ab} = 5\,. \tag{B.8}$$

Similarly, there is an $\mathfrak{so}(4,2)$ identity

$$\eta_{cc'}\big(\Sigma^{ac}\otimes\Sigma^{bc'} + \Sigma^{bc}\otimes\Sigma^{ac'}\big)_S = \frac{1}{2}\eta_{ab}\,. \tag{B.9}$$

This holds because both sides are symmetric, therefore it acts on $(0,0,1)^{\otimes_S 2} = (0,0,2)$; the resulting symmetric tensor operator Σ^{ab} would have to be in $(0,1,0)^{\otimes_S 2} = (0,2,0)+(0,0,0)$, but $(0,2,0) \notin \mathrm{End}(0,0,2)$, thus only η_{ab} can occur. We also note the following $\mathfrak{so}(4,2)$ identities:

$$\big(\epsilon_{abcdef}\Sigma^{ab}\otimes\Sigma^{cd}\big)_S = 2(\Sigma_{ef}\otimes\mathbb{1} + \mathbb{1}\otimes\Sigma_{ef}) \tag{B.10}$$

and

$$\{\Sigma^{ab},\Sigma^{cd}\}_+ = (\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc})\mathbb{1} + \frac{1}{2}\epsilon^{abcdef}\Sigma^{ef}\,. \tag{B.11}$$

In particular,

$$\epsilon^{abcdef}\Sigma^{ab}\Sigma^{cd} = 12\Sigma^{ef}\,. \tag{B.12}$$

Appendix C. Basic identities for fuzzy H_n^4

We provide the proofs for the identities given in section 3.4. First, (3.34) is obtained from

$$\begin{aligned}4X_a X^a &= r^2\bar\psi\gamma^a\psi\bar\psi'\gamma_a\psi' \\ &= r^2\bar\psi\gamma^a\gamma_a\psi + r^2\bar\psi\bar\psi'\gamma^a\otimes\gamma_a\psi\psi' \\ &= -5r^2\bar\psi\psi - r^2\bar\psi\bar\psi'\psi\psi' \\ &= -5r^2\bar\psi\psi - r^2\bar\psi(-\delta+\psi\bar\psi')\psi' \\ &= -\hat N(\hat N+4)r^2 \end{aligned} \tag{C.1}$$

using the (B.3). Similarly, (3.35) follows from

$$\begin{aligned}C^2[\mathfrak{so}(4,1)] &= \sum_{a<b\le 4}\bar\psi\Sigma^{ab}\psi\bar\psi'\Sigma_{ab}\psi' \\ &= \sum_{a<b\le 4}\left(\bar\psi\Sigma^{ab}\Sigma_{ab}\psi + \bar\psi\bar\psi'\Sigma^{ab}\otimes\Sigma_{ab}\psi\psi'\right) \\ &= \frac{5}{2}\hat N + \frac{1}{2}\bar\psi\bar\psi'\psi\psi' \\ &= \frac{1}{2}\hat N(\hat N+4)\end{aligned} \tag{C.2}$$

and (3.36) follows similarly

$$\begin{aligned}C^2[\mathfrak{so}(4,2)] &= \sum_{a<b\le 5}\bar\psi\Sigma^{ab}\psi\bar\psi'\Sigma_{ab}\psi' \\ &= \sum_{a<b\le 5}\left(\bar\psi\Sigma^{ab}\Sigma_{ab}\psi + \bar\psi\bar\psi'\Sigma^{ab}\otimes\Sigma_{ab}\psi\psi'\right) \\ &= \frac{15}{4}\hat N + \frac{3}{4}\bar\psi\bar\psi'\psi\psi' \\ &= \frac{3}{4}\hat N(\hat N+4)\end{aligned} \tag{C.3}$$

using (B.7). The identity (3.40) is obtained as

$$\begin{aligned}\epsilon_{abcde}\mathcal{M}^{ab}\mathcal{M}^{cd} &= \epsilon_{abcde}\bar\psi\Sigma^{ab}\psi\bar\psi'\Sigma^{cd}\psi' \\ &= \bar\psi\epsilon_{abcde}\Sigma^{ab}\Sigma^{cd}\psi + \bar\psi\bar\psi'\epsilon_{abcde}\Sigma^{ab}\otimes\Sigma^{cd}\psi\psi' \\ &= 12\bar\psi\Sigma^{e6}\psi + 2\bar\psi\bar\psi'(\Sigma^{e6}\otimes 1 + 1\otimes\Sigma^{e6})\psi\psi' \\ &= 12\bar\psi\Sigma^{e6}\psi + 4(\hat N\bar\psi\Sigma^{e6}\psi - \bar\psi\Sigma^{e6}\psi) \\ &= 4r^{-1}(\hat N+2)X_e = 4nr^{-1}X_e\end{aligned} \tag{C.4}$$

using (B.12) and the Euclidean identities (B.10). Finally, (3.41) is obtained from

$$\begin{aligned}\epsilon_{abcde}\mathcal{M}^{ab}X^c &= r\epsilon_{abcde}\bar\psi\Sigma^{ab}\psi\bar\psi'\Sigma^{c5}\psi' \\ &= r\bar\psi\epsilon_{abc5de}\Sigma^{ab}\Sigma^{c5}\psi + r\bar\psi\bar\psi'\epsilon_{abc5de}\Sigma^{ab}\otimes\Sigma^{c5}\psi\psi' \\ &= 3r\bar\psi\Sigma_{de}\psi + \frac{1}{2}r\bar\psi\bar\psi'(\Sigma_{de}\otimes 1 + 1\otimes\Sigma_{de})\psi\psi'\end{aligned}$$

$$= 3r\bar{\psi}\Sigma_{de}\psi + r(\hat{N}\bar{\psi}\Sigma_{de}\psi - \bar{\psi}\Sigma_{de}\psi)$$
$$= (\hat{N}+2)r\mathcal{M}_{de} = nr\mathcal{M}_{de} \tag{C.5}$$

using $\epsilon_{abc5de}\Sigma^{ab}\Sigma^{c5} = 3\Sigma^{de}$, which follows from (B.12).

C.1. Functions on $\mathcal{H}_n$, spin Casimir $\mathcal{S}^2$ and quantization

First we argue that any "reasonable" operator $\Phi \in \mathrm{End}(\mathcal{H}_n)$ (in particular any Hilbert-Schmidt (HS) operator) can be written as quantization (3.32)

$$\Phi = \mathcal{Q}(\phi) = \int d\mu\, \phi(m)\, |m\rangle \langle m| \tag{C.6}$$

of some (square-integrable, in the HS case) function $\phi(m)$ on $\mathbb{C}P^{1,2}$. To see this, assume that some operator A is orthogonal to the space spanned by (C.6), i.e. $\mathrm{Tr}(A\Phi) = 0$ for all Φ as above. Then

$$\int d\mu\, \phi(m)\, \langle m|\, A\, |m\rangle = 0 \tag{C.7}$$

for all (square-integrable, say) functions $\phi(m)$, and therefore

$$\langle m|\, A\, |m\rangle = 0 \qquad \forall\, m \tag{C.8}$$

i.e. the symbol of A vanishes. It is well-known that then A also vanishes, cf. [43]; indeed $\langle m'|A|m\rangle$ is holomorphic in $\overline{m}'$ and m, and therefore vanishes identically if it vanishes on the "diagonal" $\langle m|\, A\, |m\rangle$. The point is that coherent states $|m\rangle$ are holomorphic in m.

Now consider the spin operator (3.44)

$$\mathcal{S}^2 := C^2[\mathfrak{so}(4,1)] + \Box = \sum_{a<b\leq 4} [\mathcal{M}_{ab}, [\mathcal{M}^{ab}, \cdot]] + [X_a, [X^a, \cdot]] \tag{C.9}$$

acting on $\mathrm{End}(\mathcal{H})$. We can write this in a Lie algebra basis adapted to H^4 at the reference point $\xi \in H^4$ as follows. Let $\mathcal{M}_{ij},\ i, j = 1, 2, 3, 4$ be the $SO(4)$ generators, $E = \mathcal{M}_{05}$ the energy and

$$Z_j^{\pm} = \frac{1}{\sqrt{2}}(\mathcal{M}_{j0} \pm \mathrm{i}\mathcal{M}_{j5}) \tag{C.10}$$

be the non-compact root generators. Then

$$\begin{aligned}\mathcal{S}^2 &= (C^2[\mathfrak{so}(4)] - \delta^{ij}\mathcal{M}_{i0}\mathcal{M}_{j0}) + (\delta^{ij}\mathcal{M}_{i5}\mathcal{M}_{j5} - \mathcal{M}_{05}\mathcal{M}_{05})\\ &= C^2[\mathfrak{so}(4)] - \delta^{ij}Z_i^- Z_j^- + i\delta^{ij}(\mathcal{M}_{i0}\mathcal{M}_{j5} + \mathcal{M}_{i5}\mathcal{M}_{j0}) - E^2\\ &= C^2[\mathfrak{so}(4)] - \delta^{ij}Z_i^- Z_j^- - E^2\end{aligned} \tag{C.11}$$

using (3.37)

$$\delta^{ij}(\mathcal{M}_{i0}\mathcal{M}_{j5} + \mathcal{M}_{j5}\mathcal{M}_{i0}) = \eta_{05} = 0\,. \tag{C.12}$$

Now let $|0\rangle$ be the ground state of $\mathcal{H}_n$, which satisfies

$$Z_i^-\, |0\rangle = 0\,. \tag{C.13}$$

Then

$$\mathcal{S}^2\,|0\rangle = \Big(2\frac{n}{2}\Big(\frac{n}{2}+1\Big) - E^2\Big)|0\rangle = \Big(\frac{n}{2}+1\Big)\Big(\frac{n}{2}-1\Big)|0\rangle$$
$$=: \mathcal{S}_n^2\,|0\rangle \tag{C.14}$$

using $C^2[\mathfrak{so}(4)]|0\rangle = 2\frac{n}{2}\left(\frac{n}{2}+1\right)|0\rangle$ and $E\,|0\rangle = \left(1+\frac{n}{2}\right)|0\rangle$, see (3.28). Now consider

$$\begin{aligned}
&\mathcal{S}^2 \triangleright |0\rangle\langle 0| \\
&= 2\mathcal{S}_n^2|0\rangle\langle 0| - \mathcal{M}_{ij}|0\rangle\langle 0|\mathcal{M}^{ij} + \delta^{ij}(\mathcal{M}_{i0}|0\rangle\langle 0|\mathcal{M}_{j0} - \mathcal{M}_{i5}|0\rangle\langle 0|\mathcal{M}_{j5}) + 2E|0\rangle\langle 0|E \\
&= 2\mathcal{S}_n^2|0\rangle\langle 0| - \mathcal{M}_{ij}|0\rangle\langle 0|\mathcal{M}^{ij} + \frac{1}{2}Z_j^+|0\rangle\langle 0|Z_j^+ + \frac{1}{2}Z_j^-|0\rangle\langle 0|Z_j^- + 2E|0\rangle\langle 0|E\,,
\end{aligned}$$

noting that the cross-terms cancel. Then $Z_j^-|0\rangle = 0 = \langle 0|Z_j^+$, and for the minimal case $\mathcal{H}_0$ we have moreover $\mathcal{M}_{ij}|0\rangle = 0$ and $E|0\rangle = |0\rangle$. We conclude

$$\mathcal{S}^2 \triangleright |0\rangle\langle 0| = 2\mathcal{S}_0^2|0\rangle\langle 0| + 2|0\rangle\langle 0| = 0 \tag{C.15}$$

The same argument applies for any point $\xi \in H^4$, and (C.6) implies that $\mathrm{End}(\mathcal{H}_0)$ contains only spin $\mathcal{S}^2 = 0$ states.

For $\mathcal{H}_n$ with $n \geq 1$, we have to consider the entire $SO(4)$ orbit $g \triangleright |0\rangle\langle 0| = g\cdot|0\rangle\langle 0|\cdot g =: |m\rangle\langle m|$ over ξ where $|m\rangle = g\cdot|0\rangle$ for $g \in SO(4)$. We can express $\mathcal{M}_{ij}|m\rangle\langle m|\mathcal{M}^{ij}$ in terms of the $SO(4)$ Casimir

$$\begin{aligned}
-\mathcal{M}_{ij}|m\rangle\langle m|\mathcal{M}^{ij} &= \Big(C^2[\mathfrak{so}(4)] - 4\frac{n}{2}\Big(\frac{n}{2}+1\Big)\Big)|m\rangle\langle m| \\
&= \Big(\bigoplus_{s=0}^{n} 2s(s+1)\mathbb{1}_s - n(n+2)\Big)|m\rangle\langle m|\ .
\end{aligned} \tag{C.16}$$

Here $\bigoplus_{s=0}^{n} 2s(s+1)\mathbb{1}_s$ is the decomposition of $C^2[\mathfrak{so}(4)]$ into spin s irreps of $SU(2)$. Moreover,

$$\delta^{ij}(\mathcal{M}_{i0}|m\rangle\langle m|\mathcal{M}_{j0} - \mathcal{M}_{i5}|m\rangle\langle m|\mathcal{M}_{j5}) = 0 \tag{C.17}$$

by $SO(4)$ invariance. The same argument applies for any point $\xi \in H^4$ in (C.6). Therefore $\mathcal{S}^2$ decomposes $\mathrm{End}(\mathcal{H}_n)$ into spin s irreps as follows

$$\begin{aligned}
\mathcal{S}^2_{\mathrm{End}(\mathcal{H}_n)} &= \bigoplus_{s=0}^{n}\Big(2\mathcal{S}_n^2 + 2s(s+1)\mathbb{1}_s - n(n+2) + 2\Big(\frac{n}{2}+1\Big)^2\Big) \\
&= \bigoplus_{s=0}^{n} 2s(s+1)\mathbb{1}_s
\end{aligned} \tag{C.18}$$

which implies (3.55) and (3.56). This respects the structure of a $\mathcal{C}^0$ module, hence $\mathcal{C}$ can be viewed as a bundle over H^4 with fiber given by the space of functions on the fuzzy sphere S_n^2, with all multiplicities equal to one.

C.2. Minimal fuzzy $\mathcal{H}_{n=0}$

Consider the minireps (3.28) for the minimal case $n = 0 = j_L = j_R$. Then the lowest weight state $|\Omega\rangle = |1,0,0\rangle$ of $\mathcal{H}_0$ is the unique eigenspace with eigenvalue of $E = 1$, and there are only four non-vanishing operators with $L^+|\Omega\rangle = a^i b^j|\Omega\rangle \neq 0$, while the $SO(4)$ generators vanish on

$\Theta^{ij}|\Omega\rangle = 0,\ i,j = 1,\ldots,4$. Hence the X^a generate a 4-dimensional quantized hyperboloid H_0^4 without extra fiber. This does not seem to correspond to a coadjoint orbits of $SO(4,2)$, and the coherent states $SO(4,2)|\Omega\rangle$ form a trivial $U(1)$-bundle over H^4.

Consider the structure of $\mathcal{H}_0$ in some detail. Then $\mathrm{spec}(E_0) = \{1,2,3,\ldots\}$, and the sub-space for each given eigenspace of E_0 has the structure

$$\mathcal{H}_0|_{E_0} = (E_0)_L \otimes (E_0)_R \tag{C.19}$$

where $(m)_{L,R}$ denotes the m-dimensional irrep of $SU(2)_{L,R}$. Clearly $X^0 = rE_0$ is diagonal, while the $X^j = \frac{r}{\mathrm{i}\sqrt{2}}(Z_j^+ - Z_j^-),\ j = 1,\ldots,4$ (C.10) link the neighboring sub-spaces of (C.19) with E_0 and $E_0 \pm 1$, and similarly the $\Theta^{0j} = \frac{r^2}{\sqrt{2}}(Z_j^+ + Z_j^-)$. On the other hand, the Θ^{ij} are the $SO(4)$ generators acting within (C.19). Therefore $\langle\Omega|\Theta^{ab}|\Omega\rangle = 0$ vanishes for the local coherent states, but nevertheless $\Theta^{ab} \neq 0$ as operator. Hence H_0 is a non-commutative space which is not the quantization of a symplectic space. This is very interesting and should be investigated in more detail elsewhere. For some related mathematical results see [44].

The structure of $\mathcal{H}_n$ with $n \in \mathbb{N}$ is analogous, where (C.19) is replaced by

$$\mathcal{H}_0|_{E_0} = \Big(E_0 - \frac{n}{2}\Big)_L \otimes \Big(E_0 + \frac{n}{2}\Big)_R \ . \tag{C.20}$$

Appendix D. Auxiliary identities for semi-classical H_n^4

For any tangential $\phi_a \in \mathcal{C}$, the following identity holds:

$$x^a \eth_b \phi_a = -\phi_a \eth_b x^a = -P^{ab}\phi_a = -\phi_b \ . \tag{D.1}$$

For any tangential, divergence-free $\phi_a \in \mathcal{C}^0$, formula (3.67) yields

$$\{\theta^{ab}, \phi_b\} = r^2(x^a\eth^b - x^b\eth^a)\phi_b = -r^2 x^b \eth^a \phi_b = r^2\phi^a \ . \tag{D.2}$$

Moreover, one can verify for any ϕ that

$$\begin{aligned}\{x_c, \eth^c\phi\} &= -\frac{1}{r^2R^2}\{x_c, \theta^{cd}\{x_d,\phi\}\} \\ &= -\frac{1}{r^2R^2}\Big(\{x_c,\theta^{cd}\}\{x_d,\phi\} + \theta^{cd}\{x_c,\{x_d,\phi\}\}\Big) \\ &= \frac{1}{r^2R^2}\big(4r^2x^d\{x_d,\phi\} - \frac{1}{2}\theta^{cd}\{\theta_{cd},\phi\}\big) \\ &= 0 \ .\end{aligned} \tag{D.3}$$

Furthermore, one computes

$$\eth^d(\{x_d,\phi\}) = \frac{1}{r^2R^2}\theta_{ad}\{x^a,\{x^d,\phi\}\} = \frac{1}{2r^2R^2}\theta_{ad}\{\theta^{ad},\phi\} = 0 \tag{D.4}$$

for any $\phi \in \mathcal{C}$, and

$$\begin{aligned}\{x^b, \Box f\} &= -\{\{x^b,x^a\},\{x_a,f\}\} - \{x^a,\{\{x^b,x_a\},f\}\} - \{x^a,\{x_a,\{x^b,f\}\}\} \\ &= -\{\theta^{ba},\{x_a,f\}\} - \{x^a,\{\theta^{ba},f\}\} + \Box(\{x^b,f\}) \\ &= -\{\{\theta^{ba},x_a\},f\} - 2\{x^a,\{\theta^{ba},f\}\} + \Box(\{x^b,f\}) \\ &= -\{\{\theta^{ba},x_a\},f\} - 2r^2\{x^a,(x^b\eth_a - x^a\eth_b)f\} + \Box(\{x^b,f\})\end{aligned}$$

$$
\begin{aligned}
&= -4r^2\{x^b, f\} - 2r^2\theta^{ab}\eth_a f + \Box(\{x^b, f\}) \\
&= (\Box - 2r^2)(\{x^b, f\})
\end{aligned} \tag{D.5}
$$

for any scalar function $f \in \mathcal{C}^0$, using (D.3). Finally,

$$
\begin{aligned}
\Box(\theta^{ab}\mathcal{A}_b) &= \theta^{ab}\Box\mathcal{A}_b + (\Box\theta^{ab})\mathcal{A}_b - 2\{x^c, \theta^{ab}\}\{x^c, \mathcal{A}_b\} \\
&= \theta^{ab}\Box\mathcal{A}_b + (\Box\theta^{ab})\mathcal{A}_b - 2r^2(\eta^{ac}x^b - \eta^{bc}x^a)\{x^c, \mathcal{A}_b\} \\
&= \theta^{ab}\Box\mathcal{A}_b - 2r^2\theta^{ab}\mathcal{A}_b - 2r^2(-\{x^a, x^b\}\mathcal{A}_b - x^a\{x^c, \mathcal{A}_c\}) \\
&= \theta^{ab}\Box\mathcal{A}_b - 2r^2\theta^{ab}\mathcal{A}_b + 2r^2(\theta^{ab}\mathcal{A}_b + x^a\{x^c, \mathcal{A}_c\}) \\
&= \theta^{ab}\Box\mathcal{A}_b + 2r^2x^a\{x^c, \mathcal{A}_c\}\,.
\end{aligned} \tag{D.6}
$$

For the reducible tensor contributions (4.23), we need

$$
\begin{aligned}
\{x^b, \eth_a\phi_b\} &= \theta^{bc}\eth_c\eth_a\phi_b = \theta^{bc}\eth_a\eth_c\phi_b + \theta^{bc}[\eth_c, \eth_a]\phi_b \\
&= \eth_a(\theta^{bc}\eth_c\phi_b) - (\eth_a\theta^{bc})\eth_c\phi_b - \frac{1}{r^2R^2}\theta^{bc}\{\theta^{ca}, \phi_b\} \\
&= \eth_a\{x^b, \phi_b\} - \frac{1}{R^2}(\theta^{ac}x^b - \theta^{ab}x^c)\eth_c\phi_b + \{P^{ba}, \phi_b\} + \frac{1}{r^2R^2}\theta^{ca}\{\theta^{bc}, \phi_b\} \\
&= \eth_a\phi - \frac{1}{R^2}x^b\{x^a, \phi_b\} + \frac{1}{R^2}(x^a\{x^b, \phi_b\} - \theta^{ab}\phi_b) - \frac{1}{r^2R^2}\theta^{ca}\mathcal{I}(\phi_c) \\
&= \eth_a\phi + \frac{1}{R^2}\theta^{ab}\phi_b + \frac{1}{R^2}(x^a\phi - \theta^{ab}\phi_b) - \frac{1}{r^2R^2}\theta^{ca}\mathcal{I}(\phi_c) \\
&= \eth_a\phi + \frac{1}{R^2}x^a\phi - \frac{1}{r^2R^2}\theta^{ca}\mathcal{I}(\phi_c)
\end{aligned} \tag{D.7}
$$

using (5.16), for any tangential ϕ_a with $\phi := \{x^a, \phi_a\}$. Finally, we provide a proof for (3.65): We compute

$$
\begin{aligned}
\eth^a\eth^c\phi &= \frac{1}{r^4R^4}x_b\{\theta^{ab}, x_d\{\theta^{cd}, \phi\}\} \\
&= \frac{1}{r^4R^4}\Big(x_b\{\theta^{ab}, x_d\}\{\theta^{cd}, \phi\} + x_bx_d\{\theta^{ab}, \{\theta^{cd}, \phi\}\}\Big) \\
&= \frac{1}{r^4R^4}\Big(r^2(R^2\{\theta^{ca}, \phi\} + x^ax_d\{\theta^{cd}, \phi\}) + x_bx_d\{\theta^{ab}, \{\theta^{cd}, \phi\}\}\Big)
\end{aligned} \tag{D.8}
$$

using (3.58a). Hence

$$
\begin{aligned}
[\eth^a, \eth^c]\phi &= \frac{1}{r^4R^4}\Big(2R^2r^2\{\theta^{ca}, \phi\} + r^2(x^ax_d\{\theta^{cd}, \phi\} - (x^cx_d\{\theta^{ad}, \phi\}) \\
&\quad + x_bx_d(\{\theta^{ab}, \{\theta^{cd}, \phi\}\} - \{\theta^{cd}, \{\theta^{ab}, \phi\}\})\Big) \\
&= \frac{1}{r^4R^4}\Big(2R^2r^2\{\theta^{ca}, \phi\} + r^2(x^ax_d\{\theta^{cd}, \phi\} - x^cx_d\{\theta^{ad}, \phi\}) \\
&\quad + x_bx_d\{\{\theta^{ab}, \theta^{cd}\}, \phi\}\Big) \\
&= \frac{1}{r^4R^4}\Big(2R^2r^2\{\theta^{ca}, \phi\} + r^2\big(x^ax_d\{\theta^{cd}, \phi\} - x^cx_d\{\theta^{ad}, \phi\} \\
&\quad + x_cx_d\{\theta^{ad}, \phi\} + x_bx_a\{\theta^{bc}, \phi\} - R^2\{\theta^{ac}, \phi\}\big)\Big)
\end{aligned}
$$

$$= \frac{1}{r^4 R^4}\Big(2R^2 r^2\{\theta^{ca},\phi\} - R^2 r^2\{\theta^{ac},\phi\}\Big)$$
$$= -\frac{1}{r^2R^2}\{\theta^{ac},\phi\}\,. \tag{D.9}$$

Covariant derivative ∇. Recalling $\nabla_a\nabla_c\phi = P_{cc'}\eth_a\eth_{c'}\phi$, we obtain

$$\begin{aligned}
[\nabla_a,\nabla_c]\phi &= [\eth^a,\eth^c]\phi + \frac{1}{R^2}(x_a\eth_c - x_c\eth_a)\phi + \frac{1}{R^4}x^a x^c\left(x^d\eth_d - x^d\eth_d\right)\phi \\
&\quad + \frac{1}{r^4R^6}x_b x_d\left(x^c x^{c'}\{\theta^{ab},\{\theta^{c'd},\phi\}\} - x^a x^{a'}\{\theta^{cb},\{\theta^{a'd},\phi\}\}\right) \\
&= [\eth^a,\eth^c]\phi + \frac{1}{R^2}(x_a\eth_c - x_c\eth_a)\phi \\
&\quad + \frac{1}{r^4R^6}x_b x_d\left(-x^c x^{c'}\{\phi,\{\theta^{ab},\theta^{c'd}\}\} + x^a x^{a'}\{\phi,\{\theta^{cb},\theta^{a'd}\}\}\right) \\
&= [\eth^a,\eth^c]\phi + \frac{1}{R^2}(x_a\eth_c - x_c\eth_a)\phi \\
&= -\frac{1}{r^2R^2}\left(\{\theta^{ac},\phi\} - r^2(x_a\eth_c - x_c\eth_a)\phi\right) \\
&= -\frac{1}{r^2R^2}P^{aa'}P^{cc'}\{\theta^{a'c'},\phi\} = P^{aa'}P^{cc'}[\eth^{a'},\eth^{c'}]\phi
\end{aligned} \tag{D.10}$$

Hence, the ∇_a commute on scalar functions. For generic tensor fields $\phi_{b_1\ldots b_n}$ we have to be more careful and proceed as follows:

$$\begin{aligned}
&\nabla_a\nabla_c\phi_{b_1\ldots b_n} \\
&\quad = \nabla_a\left(P^{b_1b_1'}\cdots P^{b_nb_n'}\eth_c\phi_{b_1'\ldots b_n'}\right) \\
&\quad = P^{cc'}P^{b_1b_1'}\cdots P^{b_nb_n'}\eth_a\left(P^{b_1'b_1''}\cdots P^{b_n'b_n''}\eth_{c'}\phi_{b_1''\ldots b_n''}\right) \\
&\quad = P^{cc'}P^{b_1b_1'}\cdots P^{b_nb_n'}\eth_a\eth_{c'}\phi_{b_1'\ldots b_n'} + P^{b_1b_1'}\cdots P^{b_nb_n'}\eth_a\left(P^{b_1'b_1''}\cdots P^{b_n'b_n''}\right)\eth_c\phi_{b_1''\ldots b_n''}
\end{aligned} \tag{D.11}$$

Inspecting the second term in more detail, we arrive at

$$\begin{aligned}
\prod_{k=1}^n P^{b_kb_k'}\eth_a\left(\prod_{j=1}^n P^{b_j'b_j''}\right)\eth_c\phi_{b_1''\ldots b_n''} &= \prod_{k=1}^n P^{b_kb_k'}\sum_{j=1}^n\left((\eth_a P^{b_j'b_j''})\prod_{i\neq j}P^{b_i'b_i''}\right)\eth_c\phi_{b_1''\ldots b_n''} \\
&= \frac{1}{R^2}\sum_{j=1}^n\left(P^{ab_j}x^{b_j''}\prod_{i\neq j}P^{b_ib_i''}\right)\eth_c\phi_{b_1''\ldots b_n''} \\
&= \frac{1}{R^2}\sum_{j=1}^n\Bigg(P^{ab_j}\prod_{i\neq j}P^{b_ib_i''}\underbrace{x^{b_j''}\eth_c\phi_{b_1''\ldots b_n''}}_{-\phi_{b_1''\ldots b_{j-1}''cb_{j+1}''b_n''}}\Bigg) \\
&= -\frac{1}{R^2}\sum_{j=1}^n P^{ab_j}\phi_{b_1\ldots b_{j-1}cb_{j+1}b_n}\,.
\end{aligned} \tag{D.12}$$

Consequently, the commutator looks as follows:

$$[\nabla_a, \nabla_c]\phi_{b_1\ldots b_n} = \prod_{j=1}^{n} P^{b_j b'_j} P^{aa'} P^{cc'} \left([\eth_{a'}, \eth_{c'}]\phi_{b'_1\ldots b'_n}\right) - \frac{1}{R^2}\sum_{j=1}^{n}\left(P^{ab_j}\phi_{b_1\ldots b_{j-1}cb_{j+1}b_n} - P^{cb_j}\phi_{b_1\ldots b_{j-1}ab_{j+1}\ldots b_n}\right) . \tag{D.13}$$

With a little relabeling, we obtain

$$\begin{aligned}
[\nabla_a, \nabla_c]\phi_{b_1\ldots b_n} &= -\frac{1}{r^2R^2}\prod_{j=1}^{n} P^{b_j b'_j} P^{aa'} P^{cc'} \{\theta^{a'c'}, \phi_{b'_1\ldots b'_n}\} \\
&\quad - \frac{1}{R^2}\sum_{j=1}^{n}\left(P^{ab_j}\phi_{b_1\ldots b_{j-1}cb_{j+1}\ldots b_n} - P^{cb_j}\phi_{b_1\ldots b_{j-1}ab_{j+1}\ldots b_n}\right) \\
&= \frac{1}{R^2}\prod_{j=1}^{n} P^{b_j b'_j} P^{aa'} P^{cc'} \{m^{a'c'}, \phi_{b'_1\ldots b'_n}\} \\
&\quad - \frac{1}{R^2}\sum_{j=1}^{n}\left(P^{ab_j}P^{cd} - P^{cb_j}P^{ad}\right)\phi_{b_1\ldots b_{j-1}db_{j+1}\ldots b_n} \\
&\equiv \mathcal{R}_{ac}\phi_{b_1\ldots b_n} .
\end{aligned} \tag{D.14}$$

For an ordinary tensor field $\phi_{b_1\ldots b_n} \in \mathcal{C}^0$, the first term coincides with $\nabla_{[\eth_a,\eth_c]}$ due to (3.65), which means that curvature coincides with that of the Levi–Civita connection on H^4,

$$\begin{aligned}
\mathcal{R}_{ab}\phi_{b_1\ldots b_n} &= \sum_{j=1}^{n}\mathcal{R}_{ac;b_jd}\phi_{b_1\ldots b_{j-1}db_{j+1}\ldots b_n} , \\
\mathcal{R}_{ac;bd} &= -\frac{1}{R^2}\left(P_{ab}P_{cd} - P_{cb}P_{ad}\right) .
\end{aligned} \tag{D.15}$$

As a further check, consider $\phi_{b_1b_2} = \theta^{b_1b_2} \in \mathcal{C}^1$, where both contributions in (D.15) are non-vanishing but cancel:

$$\begin{aligned}
&[\nabla_a, \nabla_c]\theta^{b_1b_2} \\
&= -\frac{1}{r^2R^2}P^{b_1b'_1}P^{b_2b'_2}P^{aa'}P^{cc'}\{\theta^{a'c'}, \theta^{b'_1b'_2}\} \\
&\quad - \frac{1}{R^2}\left(P^{ab_1}P^{cd} - P^{cb_1}P^{ad}\right)\theta^{db_2} - \frac{1}{R^2}\left(P^{ab_2}P^{cd} - P^{cb_2}P^{ad}\right)\theta^{b_1d} \\
&= \frac{1}{R^2}P^{b_1b'_1}P^{b_2b'_2}P^{aa'}P^{cc'}\left(\eta_{a'b'_1}\theta^{c'b'_2} - \eta_{a'b'_2}\theta^{c'b'_1} - \eta_{c'b'_1}\theta^{a'b'_2} + \eta_{c'b'_2}\theta^{a'b'_1}\right) \\
&\quad - \frac{1}{R^2}\left(P^{ab_1}\theta^{cb_2} - P^{cb_1}\theta^{ab_2} + P^{ab_2}\theta^{b_1c} - P^{cb_2}\theta^{b_1a}\right) \\
&= 0
\end{aligned} \tag{D.16}$$

as it must, since $\nabla\theta^{b_1b_2} = 0$. Similarly, we can check

$$[\nabla_a, \nabla_c]x^b = -\frac{1}{r^2R^2}P^{aa'}P^{bb'}P^{cc'}\{\theta^{a'c'}, x^{b'}\} - \frac{1}{R^2}(P^{ab}P^{cd} - P^{cb}P^{ad})x^d$$

$$= \frac{1}{R^2} P^{aa'} P^{bb'} P^{cc'} \left(\theta^{a'b'} x^{c'} - \theta^{c'b'} x^{a'} \right) = 0 \,. \tag{D.17}$$

Identities for spin 1 fields. In order to derive (4.9), consider

$$\begin{aligned} 2\{x_a, \phi^{(1)}\} &= \{x_a, \mathcal{F}_{bc}\theta^{bc}\} = \mathcal{F}_{bc}\{x_a, \theta^{bc}\} + \{x_a, \mathcal{F}_{bc}\}\theta^{bc} \\ &= r^2 \mathcal{F}_{bc}(\eta_{ab}x^c - \eta_{ca}x^b) + \{x_a, \mathcal{F}_{bc}\}\theta^{bc} \\ &= r^2(\mathcal{F}_{ac}x^c - \mathcal{F}_{ba}x^b) + \{x_a, \mathcal{F}_{bc}\}\theta^{bc} \\ &= 2r^2\phi_a + \{x_a, \mathcal{F}_{bc}\}\theta^{bc} \\ &= 2r^2\phi_a + \eth_d \mathcal{F}_{bc}\theta^{ad}\theta^{bc} \,, \end{aligned} \tag{D.18}$$

where one recalls $\theta^{bc} = -r^2\mathcal{M}^{bc}$. The averaged second term can be evaluated as follows

$$\begin{aligned} \eth_d \mathcal{F}_{bc}[\theta^{ad}\theta^{bc}]_0 &= \frac{R^2 r^2}{3} \eth_d \mathcal{F}_{bc}(P_{ab}P_{cd} - P_{ac}P_{bd} + \frac{x^e}{R}\varepsilon_{adbce}) \\ &= \frac{2R^2r^2}{3} P_{ab}\eth^c \mathcal{F}_{bc} + \frac{Rr^2}{3} x^e \varepsilon_{adbce}\eth_d \mathcal{F}_{bc} \\ &= \frac{2R^2r^2}{3} P_{ab}\eth^c(\eth_c\phi_b - \eth_b\phi_c) + \frac{2Rr^2}{3} x^e\varepsilon_{adbce}\eth_d\eth_c\phi_b \\ &= \frac{2R^2r^2}{3} P_{ab}\eth^c\eth_c\phi_b - \frac{1}{3}\Big(2P_{ab}\{\theta_{bc}, \phi_c\} - \frac{x^e}{R}\varepsilon_{adcbe}\{\theta_{dc}, \phi_b\}\Big) \\ &= \frac{2R^2r^2}{3} P_{ab}\eth^c\eth_c\phi_b - \frac{2}{3} r^2\phi_a \end{aligned} \tag{D.19}$$

using (D.2), (3.65) and $\eth^c\phi_c = 0$, self-duality (D.23) and the identity (D.24). Noting that

$$x^b\eth^c\eth_c\phi_b = \eth^c\eth_c x^b\phi_b - 2\eta^{bc}\eth_c\phi_b = 0 \tag{D.20}$$

we obtain

$$P_{ab}\eth^c\eth_c\phi_b = \eth^c\eth_c\phi_a \,, \tag{D.21}$$

i.e. $\Box$ respects divergence-free tangential vector fields. Collecting all the pieces, one obtains (4.9).

Identities for spin s fields. The following identity holds for any tangential traceless divergence-free spin s field $\phi_a \in \mathcal{C}$:

$$\begin{aligned} \int P^{ac}\eth_b\phi_a\eth^b\phi_c &= \int \eth_b\phi_a\eth^b\phi^a + \frac{1}{R^2}(x^a\eth^b\phi_a)(x^c\eth_b\phi_c) \\ &= \int \eth_b\phi_a\eth^b\phi^a + \frac{1}{R^2}\phi_b\phi^b \end{aligned}$$

$$\int \eth^a f \eth_a g = -\int f \eth^a\eth_a g$$

$$\begin{aligned} \int \eth^d\phi_a\eth^a\phi_d &= -\int \phi_a\eth^d\eth^a\phi_d - \frac{4}{R^2}\int x^d\phi_a\eth^a\phi_d \\ &= -\int \phi_a[\eth^d, \eth^a]\phi_d + \frac{4}{R^2}\int \eth^a x^d\phi_a\phi_d \end{aligned}$$

$$= -\frac{1}{r^2R^2}\int \phi_a\{\theta^{ad},\phi_d\} + \frac{4}{R^2}\int \phi^a\phi_a$$

$$\int \frac{x^e}{R}\varepsilon^{abcde}\eth_b\phi_a\eth_d\phi_c = -\int \frac{x^e}{R}\varepsilon^{abcde}\phi_a\eth_b\eth_d\phi_c = \int \frac{x^e}{2r^2R^3}\varepsilon^{abcde}\phi_a\{\theta_{bd},\phi_c\} \tag{D.22}$$

and $x^a\eth_a\phi = 0$. Here $\eth\cdot\eth$ is the Euclidean Laplacian on H^4. Further, using the self-duality

$$\frac{1}{2R}\varepsilon_{adcbe}\{x^e\theta_{dc},\phi_b\} = \{\theta_{ab},\phi_b\} \tag{D.23}$$

we have the identity

$$\begin{aligned}
P_{ab}\{\theta_{bc},\phi_c\} - \frac{x^e}{2R}\varepsilon_{adcbe}\{\theta_{dc},\phi_b\} &= \frac{1}{2R}P_{aa'}\varepsilon_{a'dcbe}\theta_{dc}\{x^e,\phi_b\}\\
&= \frac{1}{2R}P_{aa'}\varepsilon_{a'dcbe}\theta_{dc}\theta^{ef}\eth_f\phi_b\\
&= -r^2P_{aa'}(g^{a'f}x^b - g^{bf}x^{a'})\eth_f\phi_b\\
&= -r^2P_{af}x^b\eth_f\phi_b = r^2\phi_a
\end{aligned} \tag{D.24}$$

using irreducibility, (3.58a) and

$$\begin{aligned}
\varepsilon_{abcde}\theta^{cd}\theta^{ef} &= \varepsilon_{abcde}\theta^{cd}\{x^e,x^f\}\\
&= \varepsilon_{abcde}\Big(\{\theta^{cd}x^e,x^f\} - x^e\{\theta^{cd},x^f\}\Big)\\
&= 2R\{\theta^{ab},x^f\} + r^2\varepsilon_{abcde}x^e(\eta_{cf}x^d - \eta_{fd}x^c)\\
&= -2r^2R(\eta_{af}x^b - \eta_{bf}x^a)\\
\varepsilon_{eadcb}\theta^{dc}\theta^{ea} &= -8r^2Rx^b
\end{aligned} \tag{D.25}$$

which is (2.22). Note that (D.24) holds for any divergence-free, tangential $\phi_b \in \mathcal{C}$.

Graviton identity. The following identity will be useful

$$\begin{aligned}
\eth_a H^{ab}[\mathcal{A}] &= \eth_a(\theta^{ca}\{\mathcal{A}_c,x^b\} + \theta^{cb}\{\mathcal{A}_c,x^a\})\\
&= (\eth_a\theta^{ca})\{\mathcal{A}_c,x^b\} + \theta^{ca}\eth_a\{\mathcal{A}_c,x^b\} + (\eth_a\theta^{cb})\{\mathcal{A}_c,x^a\}) + \theta^{cb}\eth_a\{\mathcal{A}_c,x^a\}\\
&= \eth_a\theta^{ca}\{\mathcal{A}_c,x^b\} + \{x^c,\{\mathcal{A}_c,x^b\}\} + (\eth_a\theta^{cb})\{\mathcal{A}_c,x^a\}\\
&= -\{\mathcal{A}_c,\{x^b,x^c\}\} - \{x^b,\{x^c,\mathcal{A}_c\}\} + \frac{1}{R^2}(\theta^{ac}x^b - \theta^{ab}x^c)\{\mathcal{A}_c,x^a\}\\
&= \mathcal{I}(\mathcal{A}_b) - \{x^b,\{x^c,\mathcal{A}_c\}\} - r^2(x^b\eth^c\mathcal{A}_c - x^c\eth_b\mathcal{A}_c)\\
&= \tilde{\mathcal{I}}(\mathcal{A}_b) - r^2\mathcal{A}_b - \{x^b,\{x^c,\mathcal{A}_c\}\}
\end{aligned} \tag{D.26}$$

for tangential $\mathcal{A}_a$, using (D.4) and (5.7); note that $\eth_a$ respects the projection $[\cdot]_0$.

D.1. Casimirs, positivity, and eigenvalues of $\mathcal{D}^2$

In order to show that the kinetic term is positive, we need some positivity results. A first result for spin 1 is the following. Assume that $\phi^{(1)}$ is Hermitian and determined by the tangential divergence-free vector field ϕ_a as in (4.7). Then

$$0 \leq \int \phi^{(1)}\phi^{(1)} = \int \{X_a, \phi^a\}\phi^{(1)} = -\int \phi^a\{X_a, \phi^{(1)}\}$$
$$= \frac{r^2}{3}\int \phi^a(-2 - R^2\eth\cdot\eth)\phi_a \tag{D.27}$$

This implies that $(\Box - 2r^2)\phi_a$ is positive for divergence-free square-integrable tangential tensor fields, cf. (4.39). In particular, this gives

$$\hat{\alpha}_1\phi_a = \frac{r^2}{3}(-R^2\eth\cdot\eth - 2)\phi_a, \qquad \phi_a \in \mathcal{C}^1, \qquad \hat{\alpha}_1 \geq 0\,. \tag{D.28}$$

We also observe

$$\int \phi^{(s)}\Box\phi^{(s)} \propto \int \eth_a\phi^{(s)}\eth^a\phi^{(s)} \geq 0, \qquad \phi^{(s)} \in \mathcal{C}^s \tag{D.29}$$

using (3.78), since $x\cdot\eth = 0$, i.e. $\eth\phi$ has no radial components, hence the metric is Euclidean. Therefore $\Box$ is a positive operator on any square-integrable $\phi \in \mathcal{C}^s$.

For higher spin, we need the following intertwining property of the vector fluctuations

$$r^2C^2[\mathfrak{so}(4,1)]^{(\mathrm{full})}\mathcal{A}^a[\phi^{(s)}] = -(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\mathcal{A}_a[\phi^{(s)}]$$
$$= \mathcal{A}^a[r^2C^2[\mathfrak{so}(4,1)]\phi^{(s)}]$$
$$= \mathcal{A}^a[r^2C^2_{\mathrm{full}}[\mathfrak{so}(4,1)]\phi^{(s)}_a] = \ldots$$
$$= \mathcal{A}^a[r^2C^2_{\mathrm{full}}[\mathfrak{so}(4,1)]\phi_{a_1\ldots a_s}] \tag{D.30}$$

using (5.8). The various forms on the right-hand side can be evaluated using the quadratic Casimir acting on the spin s field $\phi^{(s)}$ in its various realizations:

$$-r^2C^2\phi^{(s)} = (\Box - r^2\mathcal{S}^2)\phi^{(s)}$$
$$= (\Box - 2r^2s(s+1))\phi^{(s)} \tag{D.31}$$

and

$$-r^2C^2_{\mathrm{full}}\phi^{(s)}_a = (\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\phi^{(s)}_a$$
$$= (\Box + 2r^2(2-s) - r^2(2s(s-1)+4))\phi^{(s)}_a$$
$$= (\Box - 2r^2s^2)\phi^{(s)}_a \tag{D.32}$$

and similarly

$$-r^2C^2_{\mathrm{full}}\phi_{a_1\ldots a_s} = (\Box + 2\mathcal{I} - r^2C^2[\mathfrak{so}(4,1)]_{(s,0)})\phi_{a_1\ldots a_s}]$$
$$= (\Box + 2r^2s - r^2s(s+3))\phi_{a_1\ldots a_s}$$
$$= (\Box - r^2s(s+1))\phi_{a_1\ldots a_s} \tag{D.33}$$

because $\mathcal{S}^2 = 0$ on $\phi_{a_1\ldots a_s} \in \mathcal{C}^0$. Here we define a generalized intertwiner $\mathcal{I}$ for spin s tensor fields

$$\mathcal{I}(\phi_{a_1\ldots a_s}) := \{\theta^{aa_1}, \phi_{a_1\ldots a_s}\} + \ldots + \{\theta^{aa_s}, \phi_{a_1\ldots a_s}\} = s\{\theta^{aa_1}, \phi_{a_1\ldots a_s}\}$$
$$= r^2s\,\phi_{a_1\ldots a_s} \tag{D.34}$$

for symmetric $\phi_{a_1\ldots a_s}$, and the Casimir of $SO(4,1)$ on its indices in $(5)^{\otimes_S s}$ is

$$C^2[\mathfrak{so}(4,1)]_{(s,0)} = s(s+3)\ . \tag{D.35}$$

This is consistent with (D.38) for $s=1$. In particular, the action of $\Box$ on various realizations of the same spin s field is related as follows

$$\mathcal{A}[(\Box - 2r^2 s(s+1))\phi^{(s)}] = \mathcal{A}[(\Box - 2r^2 s^2)\phi_a^{(s)}] = \mathcal{A}[(\Box - r^2 s(s+1))\phi_{a_1\ldots a_s}]\ . \tag{D.36}$$

Now we can evaluate (D.30) for the individual spin s modes. For $\mathcal{B}^{(2)}$, we obtain

$$\begin{aligned}(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\mathcal{B}_a^{(2)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(2)}[(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\phi_a^{(s)}]\ ,\\ (\Box - 2r^2(s+1)^2\mathcal{B}_a^{(2)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(2)}[(\Box - 2r^2 s^2)\phi_a^{(s)}]\ ,\\ \mathcal{D}^2\mathcal{B}_a^{(2)}[\phi_a^{(s)}] = (\Box - 2\mathcal{I} + 4r^2)\mathcal{B}_a^{(2)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(2)}[(\Box + 2r^2 s)\phi_a^{(s)}]\ .\end{aligned} \tag{D.37}$$

For $\mathcal{B}^{(4)}$, we obtain

$$\begin{aligned}(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\mathcal{B}_a^{(4)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(4)}[(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\phi_a^{(s)}]\ ,\\ (\Box - 2r^2 s^2)\mathcal{B}_a^{(4)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(4)}[(\Box - 2r^2 s^2)\phi_a^{(s)}]\ ,\\ \mathcal{D}^2\mathcal{B}_a^{(4)}[\phi_a^{(s)}] = (\Box - 2\mathcal{I} + 4r^2)\mathcal{B}_a^{(4)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(4)}[(\Box - 2r^2 s)\phi_a^{(s)}]\ .\end{aligned} \tag{D.38}$$

For $\mathcal{B}^{(1)}$ we obtain similarly

$$\begin{aligned}(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\mathcal{B}_a^{(1)}[\phi_a^{(s)}] &= \mathcal{B}_a^{(1)}[(\Box + 2\mathcal{I} - r^2(\mathcal{S}^2+4))\phi_a^{(s)}]\ ,\\ \mathcal{D}^2\mathcal{B}_a^{(1)}[\phi^{(s)}] = (\Box + 2r^2(1+s))\mathcal{B}_a^{(1)}[\phi^{(s)}] &= \mathcal{B}_a^{(1)}[(\Box + 2r^2(3s+2))\phi_a^{(s)}]\ ,\end{aligned} \tag{D.39}$$

and finally for $\mathcal{B}^{(3)}$

$$\mathcal{D}^2\mathcal{B}_a^{(3)}[\phi_a^{(s)}] = (\Box - 2\mathcal{I} + 4r^2)\mathcal{B}_a^{(3)}[\phi_a^{(s)}] = \mathcal{B}_a^{(3)}[(\Box + 2r^2 s)\phi_a^{(s)}]\ . \tag{D.40}$$

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