

Research Article

On the vDVZ Discontinuity in Massive Conformal Gravity

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By taking the massless limit of linearized massive conformal gravity coupled to a source, we show that the theory is free from the vDVZ discontinuity. This result is confirmed when we take the massless limit of the gravitational potential of the theory, which is shown to be finite at the origin.

1. Introduction

Recently, it has been explicitly shown that the classical theory of massive conformal gravity (MCG) has a negative energy massive tensor field [1], which is a common feature of theories of gravity with fourth-order derivative terms in their actions [2]. The problem with such field is that it can produce instabilities on the classical solutions of the theory [3]. In addition, the theory may present a discontinuity in the limit as the mass of the field goes to zero.

At the quantum level, the corresponding quantum state of a negative energy field is taken to have negative energy or negative norm [4]. The requirement that the theory be renormalizable makes it necessary to choose the negative norm ghost state over the negative energy state. This choice spoils the unitarity of the S -matrix unless the position of the ghost pole is gauge-dependent [5].

In quantum MCG, the negative energy massive tensor field gives rise to a massive spin-2 ghost state. The presence of this ghost state ensures the renormalizability of the theory [1, 6]. In addition, it has recently been shown that the position of the pole of the MCG ghost state depends on the conformal gauge [7], which means that the theory is not only renormalizable but also unitary. Hence, MCG is a consistent theory of quantum gravity.

The aim of this work is to verify part of the consistency of classical MCG. Due to the extent and complexity of the calculations on the stability of the classical MCG solutions, we will leave them for future works. Here we focus on studying

the continuity of the theory in the massless limit. In Section 2 we describe MCG in the presence of matter fields. In Section 3 we analyze the particle content of the theory. In Section 4 we take the massless limit of the theory and check whether it is continuous or not. In Section 5 we evaluate the relation between the continuity of the theory and the behavior of its gravitational potential at the origin. Finally, in Section 6 we present our conclusions.

2. Massive Conformal Gravity

Let us consider the total MCG action (here we consider units in which $c = \hbar = 1$) [8]

$$\begin{aligned}
S_{\text{tot}} &= \frac{1}{k^2} \int d^4x \sqrt{-g} \left[\varphi^2 R + 6\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2m^2} C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \right] \\
&+ \int d^4x \mathcal{L}_m,
\end{aligned} \quad (1)$$

where k^2 is a constant whose value will be determined in Section 5, m is a constant with dimension of mass, φ is a scalar field called dilaton,

$$\begin{aligned}
C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} &= R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \\
&+ 2 \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right)
\end{aligned} \quad (2)$$

is the Weyl tensor squared, $R^\alpha_{\mu\beta\nu}$ is the Riemann tensor, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, and $\mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \Psi)$ is the Lagrangian density of the matter field Ψ . Besides being invariant under coordinate transformations, action (1) is also invariant under the conformal transformations

$$\begin{aligned}\tilde{g}_{\mu\nu} &= e^{2\theta(x)} g_{\mu\nu}, \\ \tilde{\varphi} &= e^{-\theta(x)} \varphi, \\ \tilde{\mathcal{L}}_m &= \mathcal{L}_m,\end{aligned}\quad (3)$$

where $\theta(x)$ is an arbitrary function of the spacetime coordinates.

It is worth noticing that under the transformations

$$\begin{aligned}\hat{g}_{\mu\nu} &= \varphi^2 g_{\mu\nu}, \\ \hat{\varphi} &= 1,\end{aligned}\quad (4)$$

action (1) takes the form

$$\begin{aligned}\hat{S}_{\text{tot}} &= \frac{1}{k^2} \int d^4x \sqrt{-\hat{g}} \left[\hat{R} - \frac{1}{2m^2} \hat{C}^{\alpha\beta\mu\nu} \hat{C}_{\alpha\beta\mu\nu} \right] \\ &+ \int d^4x \hat{\mathcal{L}}_m.\end{aligned}\quad (5)$$

In addition, the MCG line element $ds^2 = (\varphi^2 g_{\mu\nu}) dx^\mu dx^\nu$ reduces to the general relativistic line element

$$d\tilde{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu, \quad (6)$$

and the MCG geodesic equation

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{1}{\varphi} \frac{\partial \varphi}{\partial x^\rho} \left(g^{\lambda\rho} + \frac{dx^\lambda}{d\tau} \frac{dx^\rho}{d\tau} \right) = 0 \quad (7)$$

reduces to the general relativistic geodesic equation

$$\frac{d^2 x^\lambda}{d\tau^2} + \hat{\Gamma}^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (8)$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (9)$$

is the Levi-Civita connection. Thus, in the absence of matter, MCG is equivalent to the Einstein-Weyl gravity. In the presence of matter, however, the two theories are not equivalent, since the dilaton field φ reappears in the transformed matter Lagrangian density $\hat{\mathcal{L}}_m = \mathcal{L}_m(\varphi^{-2} \hat{g}_{\mu\nu}, \Psi)$.

Since the integral of the Euler density

$$E = R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \quad (10)$$

is topologically invariant, we can write (1) as

$$\begin{aligned}S_{\text{tot}} &= \frac{1}{k^2} \int d^4x \sqrt{-g} \left[\varphi^2 R + 6\partial^\mu \varphi \partial_\mu \varphi \right. \\ &\left. - \frac{1}{m^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) \right] + \int d^4x \mathcal{L}_m.\end{aligned}\quad (11)$$

By varying (11) with respect to $g^{\mu\nu}$ and φ , we obtain the MCG field equations

$$\begin{aligned}\varphi^2 G_{\mu\nu} + 6\partial_\mu \varphi \partial_\nu \varphi - 3g_{\mu\nu} \partial^\rho \varphi \partial_\rho \varphi + g_{\mu\nu} \nabla^\rho \nabla_\rho \varphi^2 \\ - \nabla_\mu \nabla_\nu \varphi^2 - \frac{1}{m^2} W_{\mu\nu} = \frac{1}{2} k^2 T_{\mu\nu},\end{aligned}\quad (12)$$

$$\left(\nabla^\mu \nabla_\mu - \frac{1}{6} R \right) \varphi = 0, \quad (13)$$

where

$$W_{\mu\nu} = \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta} - \frac{1}{2} R^{\alpha\beta} C_{\mu\alpha\nu\beta} \quad (14)$$

is the Bach tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (15)$$

is the Einstein tensor,

$$\nabla^\rho \nabla_\rho \varphi = \frac{1}{\sqrt{-g}} \partial^\rho \left(\sqrt{-g} \partial_\rho \varphi \right) \quad (16)$$

is the generally covariant d'Alembertian for a scalar field, and

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \quad (17)$$

is the matter energy-momentum tensor.

Taking the trace of (12), we find

$$6\varphi \left(\nabla^\mu \nabla_\mu - \frac{1}{6} R \right) \varphi = \frac{1}{2} k^2 T, \quad (18)$$

where $T = g^{\mu\nu} T_{\mu\nu}$. The field equations (13) and (18) require that $T = 0$, which means that MCG couples consistently only to conformally invariant matter fields. This is not a problem because it is well known that all matter fields are conformally invariant in the Standard Model of particle physics. The massless matter fields are naturally conformally invariant whereas the massive matter fields become conformally invariant after the introduction of the Higgs field.

3. The Particle Content

Using the weak-field approximations

$$g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu}, \quad (19)$$

$$\varphi = 1 + k\sigma, \quad (20)$$

and keeping only the terms of second order in the fields $h_{\mu\nu}$ and σ , we find that (11) reduces to the total linearized MCG action

$$\begin{aligned}\bar{S}_{\text{tot}} &= \int d^4x \left[\bar{\mathcal{L}}_{EH}(h_{\mu\nu}) + 2\sigma \bar{R} + 6\partial^\mu \sigma \partial_\mu \sigma \right. \\ &\left. - \frac{1}{m^2} \left(\bar{R}^{\mu\nu} \bar{R}_{\mu\nu} - \frac{1}{3} \bar{R}^2 \right) \right] + \int d^4x \left[\frac{1}{2} kh^{\mu\nu} \dot{T}_{\mu\nu}^T \right],\end{aligned}\quad (21)$$

where $\overset{\circ}{T}_{\mu\nu}^T$ is the traceless part of the flat fermion energy-momentum tensor (see the appendix),

$$\bar{R}_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \partial^\rho h_{\rho\nu} + \partial_\nu \partial^\rho h_{\rho\mu} - \partial^\rho \partial_\rho h_{\mu\nu} - \partial_\mu \partial_\nu h \right) \quad (22)$$

is the linearized Ricci tensor,

$$\bar{R} = \partial^\mu \partial^\nu h_{\mu\nu} - \partial^\mu \partial_\mu h \quad (23)$$

is the linearized scalar curvature, and

$$\begin{aligned} \bar{\mathcal{L}}_{EH}(h_{\mu\nu}) = & -\frac{1}{4} \left(\partial^\rho h^{\mu\nu} \partial_\rho h_{\mu\nu} - 2\partial^\mu h^{\nu\rho} \partial_\rho h_{\mu\nu} \right. \\ & \left. + 2\partial^\mu h_{\mu\nu} \partial^\nu h - \partial^\mu h \partial_\mu h \right) \end{aligned} \quad (24)$$

is the linearized Einstein-Hilbert Lagrangian density, with $h = \eta^{\mu\nu} h_{\mu\nu}$.

In order to obtain a second-order derivative form, we choose the method of the decomposition into oscillator variables [9] and write (21) as

$$\begin{aligned} \bar{\mathcal{S}}_{\text{tot}} = & \int d^4x \left[\frac{1}{4} h^{\mu\nu} \square q_{\mu\nu} - \frac{1}{2} h^{\mu\nu} \partial_\mu \partial^\rho q_{\rho\nu} + \frac{1}{4} h^{\mu\nu} \partial_\mu \partial_\nu q \right. \\ & + \frac{1}{4} h \partial^\mu \partial^\nu q_{\mu\nu} - \frac{1}{4} h \square q + \frac{m^2}{16} h^{\mu\nu} h_{\mu\nu} - \frac{m^2}{8} h^{\mu\nu} q_{\mu\nu} \\ & + \frac{m^2}{16} q^{\mu\nu} q_{\mu\nu} - \frac{m^2}{16} h^2 + \frac{m^2}{8} h q - \frac{m^2}{16} q^2 \\ & \left. + 2 \left(h^{\mu\nu} \partial_\mu \partial_\nu \sigma - h \square \sigma \right) - 6\sigma \square \sigma + \frac{1}{2} k h^{\mu\nu} \overset{\circ}{T}_{\mu\nu}^T \right], \end{aligned} \quad (25)$$

where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$ and $q = \eta^{\mu\nu} q_{\mu\nu}$. Varying this action with respect to $q^{\mu\nu}$ gives (the parentheses in the indices denote symmetrization)

$$\begin{aligned} q_{\mu\nu} = & h_{\mu\nu} - \frac{2}{m^2} \left[\square h_{\mu\nu} - 2\partial^\rho \partial_{(\mu} h_{\nu)\rho} + \partial_\mu \partial_\nu h \right. \\ & \left. + \frac{1}{3} \eta_{\mu\nu} \partial^\rho \partial^\sigma h_{\rho\sigma} - \frac{1}{3} \eta_{\mu\nu} \square h \right], \end{aligned} \quad (26)$$

and with this the field equations obtained from (25) are equivalent to the field equations obtained from (21). Finally, with the change of variables

$$h_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu}, \quad (27)$$

$$q_{\mu\nu} = A_{\mu\nu} - B_{\mu\nu}, \quad (28)$$

we find the action

$$\begin{aligned} \bar{\mathcal{S}}_{\text{tot}} = & \int d^4x \left[\bar{\mathcal{L}}_{EH}(A_{\mu\nu}) - \bar{\mathcal{L}}_{EH}(B_{\mu\nu}) \right. \\ & \left. + \frac{m^2}{4} (B^{\mu\nu} B_{\mu\nu} - B^2) \right. \end{aligned}$$

$$\begin{aligned} & + 2 \left(A^{\mu\nu} \partial_\mu \partial_\nu \sigma - A \square \sigma + B^{\mu\nu} \partial_\mu \partial_\nu \sigma - B \square \sigma \right) - 6\sigma \square \sigma \\ & \left. + \frac{1}{2} k A^{\mu\nu} \overset{\circ}{T}_{\mu\nu}^T + \frac{1}{2} k B^{\mu\nu} \overset{\circ}{T}_{\mu\nu}^T \right], \end{aligned} \quad (29)$$

where $A = \eta^{\mu\nu} A_{\mu\nu}$ and $B = \eta^{\mu\nu} B_{\mu\nu}$.

In order to reveal the complete particle content of (29), we consider the transformations

$$A_{\mu\nu} = A'_{\mu\nu} + \frac{1}{m} \left(\partial_\mu V_\nu + \partial_\nu V_\mu \right) + 2\eta_{\mu\nu} \sigma, \quad (30)$$

$$B_{\mu\nu} = B'_{\mu\nu} - \frac{1}{m} \left(\partial_\mu V_\nu + \partial_\nu V_\mu \right) - 2\eta_{\mu\nu} \sigma, \quad (31)$$

which gives

$$\begin{aligned} \bar{\mathcal{S}}_{\text{tot}} = & \int d^4x \left[\bar{\mathcal{L}}_{EH}(A'_{\mu\nu}) - \bar{\mathcal{L}}_{EH}(B'_{\mu\nu}) \right. \\ & + \frac{m^2}{4} (B'^{\mu\nu} B'_{\mu\nu} - B'^2) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ & + 3 \left(m^2 B' \sigma - 2m\sigma \partial_\mu V^\mu \right) \\ & - m \left(B'^{\mu\nu} \partial_\mu V_\nu - B' \partial_\mu V^\mu \right) - 6\sigma \left(\square + 2m^2 \right) \sigma \\ & \left. + \frac{1}{2} k A'^{\mu\nu} \overset{\circ}{T}_{\mu\nu}^T + \frac{1}{2} k B'^{\mu\nu} \overset{\circ}{T}_{\mu\nu}^T \right], \end{aligned} \quad (32)$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$.

Taking into account the conservation and the traceless condition of $\overset{\circ}{T}_{\mu\nu}^T$, we can see that (32) is invariant under the gauge transformations

$$A'_{\mu\nu} \longrightarrow A'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (33)$$

$$B'_{\mu\nu} \longrightarrow B'_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu, \quad (34)$$

$$V_\mu \longrightarrow V_\mu + m\chi_\mu,$$

$$B'_{\mu\nu} \longrightarrow B'_{\mu\nu} + 2m\zeta \eta_{\mu\nu},$$

$$V_\mu \longrightarrow V_\mu + \partial_\mu \zeta, \quad (35)$$

$$\sigma \longrightarrow \sigma - m\zeta,$$

where ξ_μ and χ_μ are arbitrary spacetime dependent vector fields, and ζ is an arbitrary spacetime dependent scalar field. Classically, we may impose the gauge conditions

$$\partial^\mu A'_{\mu\nu} - \frac{1}{2} \partial_\nu A' = 0, \quad (36)$$

$$\partial^\mu B'_{\mu\nu} - \frac{1}{2} \partial_\nu B' - mV_\nu = 0, \quad (37)$$

$$\partial^\mu V_\mu - m \left(\frac{1}{2} B' + 3\sigma \right) = 0, \quad (38)$$

which fix the gauge freedoms up to residual gauge parameters satisfying the subsidiary conditions

$$\square \xi_\mu = 0, \quad (39)$$

$$(\square - m^2) \chi_\mu = 0, \quad (40)$$

$$(\square - m^2) \zeta = 0. \quad (41)$$

By substituting (36)-(38) into (32) and integrating by parts, we obtain the diagonalized action

$$\begin{aligned} \bar{\mathcal{S}}_{\text{tot}}^d = \int d^4x & \left[\frac{1}{4} A'^{\mu\nu} \square A'_{\mu\nu} - \frac{1}{8} A' \square A' \right. \\ & - \frac{1}{4} B'^{\mu\nu} (\square - m^2) B'_{\mu\nu} + \frac{1}{8} B' (\square - m^2) B' \\ & - \frac{1}{2} V^\mu (\square - m^2) V_\mu - 6\sigma (\square - m^2) \sigma \\ & \left. + \frac{1}{2} k A'^{\mu\nu} T_{\mu\nu}^T + \frac{1}{2} k B'^{\mu\nu} T_{\mu\nu}^T \right], \end{aligned} \quad (42)$$

which is dynamically equivalent to action (21). This action contains a positive energy massless tensor field $A'_{\mu\nu}$, a negative energy massive tensor field $B'_{\mu\nu}$, a negative energy massive vector field V_μ , and a negative energy massive scalar field σ . In order to identify which of these fields have physical degrees of freedom, we note that $A'_{\mu\nu}$ has 10 degrees of freedom, $B'_{\mu\nu}$ has 10 degrees of freedom, V_μ has 4 degrees of freedom, and σ has 1 degree of freedom, giving a total of 25 ($= 10 + 10 + 4 + 1$) degrees of freedom. Conditions (36)-(41) reduce the degrees of freedom by 18 ($= 4 + 4 + 1 + 4 + 4 + 1$). Therefore, MCG has only 7 ($= 25 - 18$) physical degrees of freedom, 2 for the massless field $A'_{\mu\nu}$, and 5 for the massive field $B'_{\mu\nu}$.

4. The Massless Limit

In the massless limit $m \rightarrow 0$, action (32) reduces to

$$\begin{aligned} \bar{\mathcal{S}}_{\text{tot}} = \int d^4x & \left[\bar{\mathcal{L}}_{EH}(A'_{\mu\nu}) - \bar{\mathcal{L}}_{EH}(B'_{\mu\nu}) - 6\sigma \square \sigma \right. \\ & \left. + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} k A'^{\mu\nu} T_{\mu\nu}^T + \frac{1}{2} k B'^{\mu\nu} T_{\mu\nu}^T \right]. \end{aligned} \quad (43)$$

It is not difficult to see that this action is invariant under the gauge transformations

$$A'_{\mu\nu} \rightarrow A'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (44)$$

$$B'_{\mu\nu} \rightarrow B'_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu, \quad (45)$$

$$V_\mu \rightarrow V_\mu + \partial_\mu \zeta. \quad (46)$$

Thus, by imposing the gauge conditions

$$\partial^\mu A'_{\mu\nu} - \frac{1}{2} \partial_\nu A' = 0, \quad (47)$$

$$\partial^\mu B'_{\mu\nu} - \frac{1}{2} \partial_\nu B' = 0, \quad (48)$$

$$\partial^\mu V_\mu = 0 \quad (49)$$

to (43) and integrating by parts, we obtain the diagonalized action

$$\begin{aligned} \bar{\mathcal{S}}_{\text{tot}}^d = \int d^4x & \left[\frac{1}{4} A'^{\mu\nu} \square A'_{\mu\nu} - \frac{1}{8} A' \square A' - \frac{1}{4} B'^{\mu\nu} \square B'_{\mu\nu} \right. \\ & + \frac{1}{8} B' \square B' - \frac{1}{2} V^\mu \square V_\mu - 6\sigma \square \sigma + \frac{1}{2} k A'^{\mu\nu} T_{\mu\nu}^T \\ & \left. + \frac{1}{2} k B'^{\mu\nu} T_{\mu\nu}^T \right], \end{aligned} \quad (50)$$

which contains a positive energy massless tensor field $A'_{\mu\nu}$ with 10 degrees of freedom, a negative energy massless tensor field $B'_{\mu\nu}$ with 10 degrees of freedom, a negative energy massless vector field V_μ with 4 degrees of freedom, and a negative energy massless scalar field σ with 1 degree of freedom.

The gauge conditions (47)-(49) fix the gauge freedoms up to residual gauge parameters satisfying the subsidiary conditions

$$\square \xi_\mu = 0, \quad (51)$$

$$\square \chi_\mu = 0, \quad (52)$$

$$\square \zeta = 0. \quad (53)$$

Conditions (47)-(49) and (51)-(53) reduce the 25 degrees of freedom of the MCG massless limit by 18, giving a total of 7 physical degrees of freedom, 2 for the massless tensor field $A'_{\mu\nu}$, 2 for the massless tensor field $B'_{\mu\nu}$, 2 for the massless vector field V_μ , and 1 for the massless scalar field σ . Hence, the number of physical degrees of freedom is preserved in the massless limit of the theory. Two of the five degrees of freedom of the massive tensor field go into the massless vector field, and one goes into the massless scalar field. Since the scalar and vector fields do not couple to the source, MCG is free of the van Dam-Veltman-Zakharov (vDVZ) discontinuity [10, 11].

It is worth noticing that MCG differs from theories of gravity with only a massive tensor field, for which general relativity (GR) should be recovered in the massless limit. Usually it is the absence of this recovery that is associated with the vDVZ discontinuity. However, the real meaning of the vDVZ discontinuity is that the massless limit of a massive theory is not equivalent to the massless theory, which violates the physical continuity principle [12]. That is why it is so important for a massive theory to be free from the vDVZ discontinuity.

Recently, it was proposed that the vDVZ discontinuity of a local gravitational theory is closely related to the finiteness of the gravitational potential at the origin [13]. If this proposal is valid, then the MCG potential should have a singularity at the origin, which we will show to be not true in the next section.

5. The Newtonian Singularity

The variation of (42) with respect to $A'^{\mu\nu}$ and $B'^{\mu\nu}$ gives the field equations

$$\square \left(A'_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} A' \right) = -k T_{\mu\nu}^T, \quad (54)$$

$$(\square - m^2) \left(B'_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} B' \right) = k T_{\mu\nu}^T. \quad (55)$$

Taking the traces of (54) and (55) and replacing them back, we obtain

$$\square A'_{\mu\nu} = -k \overset{\circ}{T}_{\mu\nu}^T, \quad (56)$$

$$(\square - m^2) B'_{\mu\nu} = k \overset{\circ}{T}_{\mu\nu}^T, \quad (57)$$

whose general solutions are given by

$$A'_{\mu\nu} = k \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip \cdot x}}{p^2} \overset{\circ}{T}_{\mu\nu}^T(p), \quad (58)$$

$$B'_{\mu\nu} = -k \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip \cdot x}}{p^2 + m^2} \overset{\circ}{T}_{\mu\nu}^T(p), \quad (59)$$

where

$$\overset{\circ}{T}_{\mu\nu}^T(p) = \int d^4 x e^{-ip \cdot x} \overset{\circ}{T}_{\mu\nu}^T(x) \quad (60)$$

is the Fourier transform of the traceless part of the source.

In the case of a point particle source with mass M at rest at the origin, for which $\overset{\circ}{T}_{\mu\nu}^f(x) = M \delta_\mu^0 \delta_\nu^0 \delta^3(\mathbf{x})$, we have

$$\overset{\circ}{T}_{\mu\nu}^T = \left(M \delta_\mu^0 \delta_\nu^0 + \frac{1}{4} \eta_{\mu\nu} M \right) \delta^3(\mathbf{x}), \quad (61)$$

where we consider $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$. Substituting the 00 component of (61) into (58) and (59), we find the general solutions

$$A'_{00} = k \frac{3M}{4} \int \frac{d^3 \mathbf{p}}{(2\pi)^4} \frac{e^{ip \cdot x}}{p^2} = k \frac{3M}{4} \frac{1}{4\pi r}, \quad (62)$$

$$B'_{00} = -k \frac{3M}{4} \int \frac{d^3 \mathbf{p}}{(2\pi)^4} \frac{e^{ip \cdot x}}{p^2 + m^2} = -k \frac{3M}{4} \frac{e^{-mr}}{4\pi r}. \quad (63)$$

It then follows from (27), (30), and (31) that

$$h_{00} = A'_{00} + B'_{00} = k \frac{3M}{16\pi r} (1 - e^{-mr}). \quad (64)$$

Finally, noting that $2\phi = -kh_{00}$, we find the MCG potential

$$\phi(r) = -\frac{GM}{r} (1 - e^{-mr}), \quad (65)$$

where we chose $k^2 = 32\pi G/3$ in order for (65) to agree with the Newtonian potential in the limit where m tends to infinity. It is not difficult to see that (65) is finite at the origin, which is a necessary condition for the renormalizability of the theory [14]. In addition, the massless limit provides a zero-order term of $e^{-mr} \approx 1$ in the MCG potential, which confirms that the theory is free of the vDVZ discontinuity.

6. Final Remarks

We have found that MCG contains only two linearized fields with physical degrees of freedom: the usual positive energy massless tensor field and a negative energy massive tensor

field. In the massless limit, the number of physical degrees of freedom of the theory remains the same, with the massive tensor field being decomposed into massless tensor, vector, and scalar fields. The absence of coupling between the vector and scalar fields and the source makes the theory free of the vDVZ discontinuity. In addition, it was shown that the MCG potential is singularity free, which indicates that the vDVZ discontinuity is not directly related to the finiteness of the gravitational potential at the origin.

Appendix

Linearized MCG Matter Lagrangian Density

As we saw in Section 2, the matter Lagrangian density has to be conformally invariant in MCG. For simplicity, let us consider the Lagrangian density for a fermion field ψ conformally coupled to the gravitational field $g_{\mu\nu}$, which is given by

$$\begin{aligned} \mathcal{L}_m = & -\sqrt{-g} \left[\frac{1}{12} S^2 R + \frac{1}{2} \partial^\mu S \partial_\mu S + \frac{1}{4!} \lambda S^4 \right. \\ & \left. + \frac{i}{2} (\bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi) + \mu S \bar{\psi} \psi \right], \end{aligned} \quad (A.1)$$

where S is a real Higgs scalar field, λ and μ are dimensionless coupling constants, $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint fermion field, $D_\mu = \partial_\mu + [\gamma^\nu, \partial_\nu \gamma_\nu]/8 - [\gamma^\nu, \gamma_\nu] \Gamma_{\mu\nu}^\lambda/8$, and γ^μ are the general relativistic Dirac matrices, which satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

The variation of (A.1) with respect to the matter fields S , $\bar{\psi}$, and ψ gives

$$\left(\nabla^\mu \nabla_\mu - \frac{1}{6} R \right) S + \frac{1}{6} \lambda S^3 + \mu \bar{\psi} \psi = 0, \quad (A.2)$$

$$i \gamma^\mu D_\mu \psi + \mu S \psi = 0, \quad (A.3)$$

$$i D_\mu \bar{\psi} \gamma^\mu - \mu S \bar{\psi} = 0. \quad (A.4)$$

By substituting (A.1) into (17) and using the matter field equations (A.2)-(A.4), we can write the MCG matter energy-momentum tensor in the form

$$\begin{aligned} T_{\mu\nu} = & T_{\mu\nu}^f - \frac{1}{4} g_{\mu\nu} T^f + \frac{1}{6} S^2 \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) \\ & + \frac{2}{3} \nabla_\mu S \nabla_\nu S - \frac{1}{6} g_{\mu\nu} \nabla^\rho S \nabla_\rho S - \frac{1}{3} S \nabla_\mu \nabla_\nu S \\ & + \frac{1}{12} g_{\mu\nu} S \nabla^\rho \nabla_\rho S, \end{aligned} \quad (A.5)$$

where

$$\begin{aligned} T_{\mu\nu}^f = & \\ = & \frac{i}{4} (\bar{\psi} \gamma_\mu D_\nu \psi - D_\nu \bar{\psi} \gamma_\mu \psi + \bar{\psi} \gamma_\nu D_\mu \psi - D_\mu \bar{\psi} \gamma_\nu \psi) \end{aligned} \quad (A.6)$$

is the fermion energy-momentum tensor, and

$$T^f = \frac{i}{2} (\bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi) \quad (\text{A.7})$$

is the trace of the fermion energy-momentum tensor.

With the help of the conformal transformations

$$\begin{aligned} \tilde{g}_{\mu\nu} &= e^{2\theta(x)} g_{\mu\nu}, \\ \tilde{S} &= e^{-\theta(x)} S, \\ \tilde{\psi} &= e^{-3\theta(x)/2} \psi, \\ \tilde{\gamma}^\mu &= e^{-\theta(x)} \gamma^\mu, \end{aligned} \quad (\text{A.8})$$

it is possible to verify that (A.1) is conformally invariant. Using this symmetry, we can impose the unitary gauge $S = S_0$, where S_0 is a spontaneously broken constant expectation value for the Higgs field. In this case, (A.5) reduces to

$$T_{\mu\nu} = T_{\mu\nu}^f - \frac{1}{4} g_{\mu\nu} T^f + \frac{1}{6} S_0^2 \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right). \quad (\text{A.9})$$

It is not difficult to see that both (A.5) and (A.9) are traceless, which means that the traceless condition of the MCG matter energy-momentum is independent of the gauge.

It is well known that the weak-field limit of (17) leads to the linearized matter Lagrangian density

$$\overline{\mathcal{L}}_m = \frac{1}{2} k h^{\mu\nu} \overset{\circ}{T}_{\mu\nu}, \quad (\text{A.10})$$

where $\overset{\circ}{T}_{\mu\nu}$ is the flat matter energy-momentum tensor. Taking the flat limit of (A.9) and substituting into (A.10), we find the linearized MCG matter Lagrangian density

$$\overline{\mathcal{L}}_m = \frac{1}{2} k h^{\mu\nu} \overset{\circ}{T}_{\mu\nu}^T, \quad (\text{A.11})$$

where

$$\overset{\circ}{T}_{\mu\nu}^T = \overset{\circ}{T}_{\mu\nu}^f - \frac{1}{4} \eta_{\mu\nu} \overset{\circ}{T}^f \quad (\text{A.12})$$

is the traceless part of the flat fermion energy-momentum tensor.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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