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Gauge (in)dependence and background field formalism

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ABSTRACT

It is shown that the gauge invariance and gauge dependence properties of effective action for Yang-Mills theories should be considered as two independent issues in the background field formalism. Application of this formalism to formulate the functional renormalization group approach is discussed. It is proven that there is a possibility to construct the corresponding average effective action invariant under the gauge transformations of background vector field. Nevertheless, being gauge invariant this action remains gauge dependent on-shell.

1. Introduction

It is well-known fact that the gauge symmetry of an initial action is broken on quantum level because of the gauge fixing procedure in process of quantization. Generating functional of vertex functions (effective action) being main quantity in quantum field theory depends on gauges [1–4]. This dependence has a special form and disappears on-shell [5,6]. In its turn it allows to have a physical interpretation of results obtained on quantum level.

The background field method [7–9] presents a reformulation of quantization procedure for Yang-Mills theories allowing to work with the effective action invariant under the gauge transformations of background fields and to reproduce all usual physical results by choosing a special background field condition [9,10]. Various aspects of quantum properties of gauge theories have been successfully studied in this technique [11–20]. Application of the background field method simplifies essentially calculations of Feynman diagrams in gauge theories (among recent applications of this approach see, for example, [21–25]). The gauge dependence problem in this method remains very important matter although it does not discuss because standard considerations are restricted by the background field gauge condition and by the invariance of generating functionals of Green functions under gauge transformations of background fields.

In the present paper we study the gauge dependence of generating functionals of Green functions in the background field formalism for Yang-Mills theories in class of gauges depending on gauge and background vector fields. The background field gauge condition belongs them as a special choice. We prove that the gauge invariance can be achieved if the gauge fixing functions satisfy a tensor transformation law and are linear in gauge fields. We consider the gauge dependence and gauge invariance problems within the background field formalism as two independent ones. To support this point of view we analyze the functional renormalization group (FRG) approach [26,27] in the background field formalism. We find restrictions on tensor structure of the regulator functions which allow to construct a gauge invariant average effective action. Nevertheless, being gauge invariant this action remains a gauge dependent quantity on-shell making impossible a physical interpretation of results obtained for gauge theories.

The paper is organized as follows. Section 2 is devoted to description of the background field formalism in gauges more general than the usual background field gauge condition, to prove the gauge independence of vacuum functional and to study symmetry properties of the effective action. In Section 3 we analyze the gauge invariance of background average effective action for the FRG approach and find restrictions on regulator functions admitting this invariance. In section 4 we prove the gauge dependence of vacuum functional (and therefore S-matrix) for the FRG approach. In section 5 concluding remarks are given.

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In the paper the DeWitt's condensed notations are used [28]. We employ the notation $\varepsilon(A)$ for the Grassmann parity and the $\mathrm{gh}(A)$ for the ghost number of any quantity A. All functional derivatives are taken from the left. The functional right derivatives with respect to fields are marked by special symbol "$\leftarrow$".

2. Background field formalism for Yang-Mills theories

We start with a gauge theory of non-abelian vector fields $A^\alpha_\mu(x)$ $(\varepsilon(A^\alpha_\mu(x))=0,\ \mathrm{gh}(A^\alpha_\mu(x))=0)$ formulated in the Minkowski space-time of arbitrary dimension with the action

$$\mathcal{S}_{YM}(A)=\int dx\Big(-\frac{1}{4}G^\alpha_{\mu\nu}(A(x))G^\alpha_{\mu\nu}(A(x))\Big), \tag{2.1}$$

where the notation

$$G^\alpha_{\mu\nu}(A(x))=\partial_\mu A^\alpha_\nu(x)-\partial_\nu A^\alpha_\mu(x)+gf^{\alpha\beta\gamma}A^\beta_\mu(x)A^\gamma_\nu(x), \tag{2.2}$$

is used. In relation (2.2) $f^{\alpha\beta\gamma}$ are structure coefficients of a compact simple gauge Lie group and g is a gauge interaction constant. The action (2.1) is invariant under gauge transformations with arbitrary gauge functions $\omega_\alpha(x)$,

$$\begin{aligned}&\delta_\omega\mathcal{S}_{YM}(A)=0,\\ &\delta_\omega A^\alpha_\mu(x)=\big(\partial_\mu\delta_{\alpha\beta}+gf^{\alpha\sigma\beta}A^\sigma_\mu(x)\big)\omega_\beta(x)=D^{\alpha\beta}_\mu(A(x))\omega_\beta(x).\end{aligned} \tag{2.3}$$

In the background field formalism [7–9] the gauge field $A^\alpha_\mu(x)$ appearing in classical action (2.1), is replaced by $A^\alpha_\mu(x)+\mathcal{B}^\alpha_\mu(x)$,

$$\mathcal{S}_{YM}(A)\ \rightarrow\ \mathcal{S}_{YM}(A+\mathcal{B}), \tag{2.4}$$

where $\mathcal{B}^\alpha_\mu(x)$ is considered as an external field. The action $\mathcal{S}_{YM}(A+\mathcal{B})$ obeys obviously the gauge invariance,[1]

$$\delta_\omega\mathcal{S}_{YM}(A+\mathcal{B})=0,\quad \delta_\omega A^\alpha_\mu=D^{\alpha\beta}_\mu(A+\mathcal{B})\omega_\beta. \tag{2.5}$$

The corresponding Faddeev-Popov action $S_{FP}=S_{FP}(\phi,\mathcal{B})$ has the form [29][2]

$$\begin{aligned}S_{FP}=\mathcal{S}_{YM}(A+\mathcal{B})+\int dx\Big[\overline{C}^\alpha\Big(\chi_\alpha(A,\mathcal{B})\frac{\overleftarrow{\delta}}{\delta A^\beta_\mu}\Big)D^{\beta\gamma}_\mu(A+\mathcal{B})C^\gamma\\ +B^\alpha\chi_\alpha(A,\mathcal{B})\Big],\end{aligned} \tag{2.6}$$

where $\chi_\alpha(A,\mathcal{B})$ are functions lifting the degeneracy of the Yang-Mills action, $\phi=\{\phi^i\}$ is the set of all fields $\phi^i=(A^\alpha_\mu,B^\alpha,C^\alpha,\overline{C}^\alpha)$ $(\varepsilon(\phi^i)=\varepsilon_i)$ with the Faddeev-Popov ghost and anti-ghost fields $C^\alpha,\overline{C}^\alpha$ $(\varepsilon(C^\alpha)=\varepsilon(\overline{C}^\alpha)=1,\ \mathrm{gh}(C^\alpha)=-\mathrm{gh}(\overline{C}^\alpha)=1)$, respectively, and the Nakanishi-Lautrup auxiliary fields B^α $(\varepsilon(B^\alpha)=0,$ $\mathrm{gh}(B^\alpha)=0)$. A standard choice of $\chi_\alpha(A,\mathcal{B})$ corresponding to the background field gauge condition [9], reads

$$\chi_\alpha(A,\mathcal{B})=D^{\alpha\beta}_\mu(\mathcal{B})A^\beta_\mu. \tag{2.7}$$

In what follows the specific form of $\chi_\alpha(A,\mathcal{B})$ is not essential for all results obtained but the property of linearity of these functions with respect to fields A^α_μ plays a crucial role in the background-field formalism.

The action (2.6) is invariant under global supersymmetry (BRST symmetry) [30,31]

$$\begin{aligned}&\delta_B A^\alpha_\mu=D^{\alpha\beta}_\mu(A+\mathcal{B})C^\beta\mu,\quad \delta_B C^\alpha=\frac{g}{2}f^{\alpha\beta\gamma}C^\beta C^\gamma\mu,\\ &\delta_B\overline{C}^\alpha=B^\alpha\mu,\quad \delta_B B^\alpha=0,\end{aligned} \tag{2.8}$$

where μ is a constant anti-commuting parameter or, in short,

$$\delta_B\phi^i=R^i(\phi,\mathcal{B})\mu,\quad \varepsilon(R^i(\phi,\mathcal{B}))=\varepsilon_i+1, \tag{2.9}$$

where

$$R^i(\phi,\mathcal{B})=\big(D^{\alpha\beta}_\mu(A+\mathcal{B})C^\beta,\ 0\ ,\ \frac{g}{2}f^{\alpha\beta\gamma}C^\beta C^\gamma,\ B^\alpha\big). \tag{2.10}$$

Introducing the gauge fixing functional $\Psi=\Psi(\phi,\mathcal{B})$,

$$\Psi=\int dx\,\overline{C}^\alpha\chi_\alpha(A,\mathcal{B}), \tag{2.11}$$

the action (2.6) rewrites in the form

$$\begin{aligned}&S_{FP}(\phi,\mathcal{B})=\mathcal{S}_{YM}(A+\mathcal{B})+\Psi(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B}),\\ &\mathcal{S}_{YM}(A+\mathcal{B})\hat{R}(\phi,\mathcal{B})=0,\end{aligned} \tag{2.12}$$

where

$$\hat{R}(\phi,\mathcal{B})=\int dx\,\frac{\overleftarrow{\delta}}{\delta\phi^i}R^i(\phi,\mathcal{B}) \tag{2.13}$$

is the generator of BRST transformations. Due to the nilpotency property of $\hat{R}$, $\hat{R}^2=0$, the BRST symmetry of S_{FP} follows from the presentation (2.12) immediately,

$$S_{FP}(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B})=0. \tag{2.14}$$

The generating functional of Green functions in the background field method is defined in the form of functional integral

$$Z(J,\mathcal{B})=\int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[S_{FP}(\phi,\mathcal{B})+J\phi\big]\Big\}=\exp\Big\{\frac{i}{\hbar}W(J,\mathcal{B})\Big\}, \tag{2.15}$$

where $W(J,\mathcal{B})$ is the generating functional of connected Green functions. In (2.15) the notations

$$J\phi=\int dx J_i(x)\phi^i(x),\quad J_i(x)=(J^\alpha_\mu(x),\ J^{(B)}_\alpha(x),\ \overline{J}_\alpha(x),\ J_\alpha)(x) \tag{2.16}$$

are used and $J_i(x)$ $\big(\varepsilon(J_i(x))=\varepsilon_i,\ \mathrm{gh}(J_i(x))=\mathrm{gh}(\phi^i(x))\big)$ are external sources to fields $\phi^i(x)$.

Let $Z_\Psi(\mathcal{B})$ be the vacuum functional which corresponds to the choice of gauge fixing functional (2.11) in the presence of external fields $\mathcal{B}$,

$$\begin{aligned}Z_\Psi(\mathcal{B})&=\int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[\mathcal{S}_{YM}(A+\mathcal{B})+\Psi(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B})\big]\Big\}\\ &=\int d\phi\ \exp\Big\{\frac{i}{\hbar}S_{FP}(\phi,\mathcal{B})\Big\}.\end{aligned} \tag{2.17}$$

In turn, let $Z_{\Psi+\delta\Psi}$ be the vacuum functional corresponding to a gauge fixing functional $\Psi(\phi,\mathcal{B})+\delta\Psi(\phi,\mathcal{B})$,

$$Z_{\Psi+\delta\Psi}(\mathcal{B})=\int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[S_{FP}(\phi,\mathcal{B})+\delta\Psi(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B})\big]\Big\}. \tag{2.18}$$

[1] In what follows we will omit the space-time argument x of fields and gauge parameters when this does not lead to misunderstandings in the formulas and relations.

[2] The action (2.6) is written in so-called singular gauge fixing. Non-singular gauge fixing corresponds to addition in the right-hand side of (2.6) the term $\int dx B^\alpha g_{\alpha\beta}B^\beta$ where $g_{\alpha\beta}=g_{\beta\alpha}$ are elements of a constant invertible matrix. The term is invariant under BRST transformations and does not spoil the renormalization properties of the theory under consideration.

Here, $\delta\Psi(\phi,\mathcal{B})$ is an arbitrary infinitesimal odd functional which may, in general, have a form differing on (2.11). Making use of the change of variables ϕ^i in the form of BRST transformations (2.9) but with replacement of the constant parameter μ by the following functional

$$\mu=\mu(\phi,\mathcal{B})=\frac{i}{\hbar}\delta\Psi(\phi,\mathcal{B}), \tag{2.19}$$

and taking into account that the Jacobian of transformations is equal to

$$J=\exp\{-\mu(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B})\}, \tag{2.20}$$

we find the gauge independence of the vacuum functional

$$Z_\Psi(\mathcal{B})=Z_{\Psi+\delta\Psi}(\mathcal{B}). \tag{2.21}$$

The property (2.21) was a reason to omit the label Ψ in the definition of generating functionals (2.15). In deriving (2.21) the relation

$$(-1)^{\varepsilon_i}\frac{\partial}{\partial\phi^i}R^i(\phi,\mathcal{B})=0, \tag{2.22}$$

was used. It holds due to the antisymmetry property of structure constants, $f^{\alpha\beta\gamma}=-f^{\beta\alpha\gamma}$. In turn, the property (2.21) means that due to the equivalence theorem [32] the physical S-matrix does not depend on the gauge fixing.

The vacuum functional $Z(\mathcal{B})=Z(J=0,\mathcal{B})$ obeys the very important property of gauge invariance with respect to gauge transformations of external fields,

$$\delta_\omega\mathcal{B}^\alpha_\mu=D^{\alpha\beta}_\mu(\mathcal{B})\omega_\beta,\quad \delta_\omega Z(\mathcal{B})=0. \tag{2.23}$$

It means the gauge invariance of functional $W(\mathcal{B})=W(J=0,\mathcal{B})$, $\delta_\omega W(\mathcal{B})=0$, as well. The proof is based on using the change of variables in the functional integral (2.17) of the following form

$$\begin{aligned}&\delta_\omega A^\alpha_\mu=gf^{\alpha\gamma\beta}A^\gamma_\mu\omega_\beta,\quad \delta_\omega C^\alpha=gf^{\alpha\gamma\beta}C^\gamma\omega_\beta,\\ &\delta_\omega\overline{C}^\alpha=gf^{\alpha\gamma\beta}\overline{C}^\gamma\omega_\beta,\quad \delta_\omega B^\alpha=gf^{\alpha\gamma\beta}B^\gamma\omega_\beta\end{aligned} \tag{2.24}$$

taking into account that the Jacobian of transformations (2.24) is equal to a unit, and assuming the transformation law of gauge fixing functions χ_α according to

$$\delta_\omega\chi_\alpha(A,\mathcal{B})=gf^{\alpha\gamma\beta}\chi_\gamma(A,\mathcal{B})\omega_\beta, \tag{2.25}$$

which is fulfilled explicitly for the background field gauge condition (2.7). In particular, it can be argued the invariance of $S_{FP}(\phi,\mathcal{B})$ under combined gauge transformations (2.23) and (2.24)

$$\delta_\omega S_{FP}(\phi,\mathcal{B})=0. \tag{2.26}$$

The Slavnov-Taylor identity for the generating functional of Green functions is derived in standard manner,

$$\int dx\,J_iR^i\Big(\frac{\hbar}{i}\frac{\delta}{\delta J},\mathcal{B}\Big)Z(J,\mathcal{B})=0, \tag{2.27}$$

as consequence of the BRST symmetry of S_{FP} (2.14) on the quantum level. In terms of generating functional of connected Green functions, $W(J,\mathcal{B})$, the identity (2.27) rewrites as

$$\int dx\,J_iR^i\Big(\frac{\delta W(J,\mathcal{B})}{\delta J}+\frac{\hbar}{i}\frac{\delta}{\delta J},\mathcal{B}\Big)\cdot 1=0. \tag{2.28}$$

The generating functional of vertex functions (effective action), $\Gamma=\Gamma(\Phi,\mathcal{B})$, is defined in a standard form through the Legendre transformation of $W(J,\mathcal{B})$,

$$\begin{aligned}&\Gamma(\Phi,\mathcal{B})=W(J,\mathcal{B})-\int dx\,J_i\Phi^i,\quad \Phi^i=\frac{\delta W(J)}{\delta J_i},\\ &\Phi^i=(\mathcal{A}^\alpha_\mu,\Phi^\alpha_{(B)},\mathcal{C}^\alpha,\overline{\mathcal{C}}^\alpha),\end{aligned} \tag{2.29}$$

so that

$$\Gamma(\Phi,\mathcal{B})\frac{\overleftarrow{\delta}}{\delta\Phi^i}=-J_i. \tag{2.30}$$

The Ward identity (2.28) rewrites for $\Gamma(\Phi,\mathcal{B})$ in the form

$$\Gamma(\Phi,\mathcal{B})\widehat{\widehat{R}}(\Phi,\mathcal{B})=0, \tag{2.31}$$

where

$$\widehat{\widehat{R}}(\Phi,\mathcal{B})=\int dx\frac{\overleftarrow{\delta}}{\delta\Phi^i}\overline{R}^i(\Phi,\mathcal{B}),\quad \overline{R}^i(\Phi,\mathcal{B})=R^i(\hat{\Phi},\mathcal{B})\cdot 1, \tag{2.32}$$

can be considered as the generator of quantum BRST transformations. In relation (2.32) the notations

$$\hat{\Phi}^i(x)=\Phi^i(x)+i\hbar\int dy(\Gamma''^{-1})^{ij}(\Phi,\mathcal{B})(x,y)\frac{\overrightarrow{\delta}}{\delta\Phi^j(y)}, \tag{2.33}$$

are used. In turn the matrix $(\Gamma''^{-1})^{ij}(x,y)=(\Gamma''^{-1})^{ij}(\Phi,\mathcal{B})(x,y)$ is inverse to the matrix of second derivatives of effective action,

$$(\Gamma'')_{ij}(\Phi,\mathcal{B})(x,y)=\frac{\overrightarrow{\delta}}{\delta\Phi^i(x)}\Big(\Gamma(\Phi,\mathcal{B})\frac{\overleftarrow{\delta}}{\delta\Phi^j(y)}\Big), \tag{2.34}$$

$$\int dz(\Gamma''^{-1})^{ik}(x,z)(\Gamma'')_{kj}(z,y)=\delta^i_j\delta(x-y). \tag{2.35}$$

The Ward identity (2.31) can be interpreted as the invariance of effective action $\Gamma(\Phi,\mathcal{B})$ under the *quantum* BRST transformations of Φ^i with generators $\bar{R}^i(\Phi,\mathcal{B})$.

Notice that in the case of anomaly-free theories and a regularization preserving the gauge invariance, one can prove in the standard manner [6] (see also [10]) that the renormalized action $S_{FP,ren}(\phi,\mathcal{B})$ and the renormalized effective action $\Gamma_{ren}(\Phi,\mathcal{B})$ satisfy the same equations (2.14) and (2.31) with the corresponding nilpotent operators $\hat{R}_{ren}(\phi,\mathcal{B})$ and $\widehat{\widehat{R}}_{ren}(\Phi,\mathcal{B})$, respectively.

The invariance of S_{FP} (2.26) means that the functional $Z(J,\mathcal{B})$ is invariant

$$\begin{aligned}&Z(J,\mathcal{B})\int dx\frac{\overleftarrow{\delta}}{\delta\mathcal{B}^\alpha_\mu}D^{\alpha\beta}_\mu(\mathcal{B})\omega_\beta\\ &=gf^{\alpha\gamma\beta}\omega_\beta\int dx\Big(J^\alpha_\mu\frac{\delta}{\delta J^\gamma_\mu}+J_\alpha\frac{\delta}{\delta J_\gamma}+\overline{J}_\alpha\frac{\delta}{\delta\overline{J}_\gamma}+J^{(B)}_\alpha\frac{\delta}{\delta J^{(B)}_\gamma}\Big)\\ &\quad\times Z(J,\mathcal{B}),\end{aligned} \tag{2.36}$$

under the gauge transformations of the background vector field $\mathcal{B}$ (2.23) and simultaneously the tensor transformations of sources

$$\begin{aligned}&\delta_\omega J^\alpha_\mu=gf^{\alpha\gamma\beta}J^\gamma_\mu\omega_\beta,\ \delta_\omega\overline{J}_\alpha=gf^{\alpha\gamma\beta}\overline{J}_\gamma\omega_\beta,\\ &\delta_\omega J_\alpha=gf^{\alpha\gamma\beta}J_\gamma\omega_\beta,\ \delta_\omega J^{(B)}_\alpha=gf^{\alpha\gamma\beta}J^{(B)}_\gamma\omega_\beta.\end{aligned} \tag{2.37}$$

In its turn the functional $W(J,\mathcal{B})$ obeys the same symmetry property as well,

$$\begin{aligned}&W(J,\mathcal{B})\int dx\frac{\overleftarrow{\delta}}{\delta\mathcal{B}^\alpha_\mu}D^{\alpha\beta}_\mu(\mathcal{B})\omega_\beta\\ &=gf^{\alpha\gamma\beta}\omega_\beta\int dx\Big(J^\alpha_\mu\frac{\delta}{\delta J^\gamma_\mu}+J_\alpha\frac{\delta}{\delta J_\gamma}+\overline{J}_\alpha\frac{\delta}{\delta\overline{J}_\gamma}+J^{(B)}_\alpha\frac{\delta}{\delta J^{(B)}_\gamma}\Big)\\ &\quad\times W(J,\mathcal{B}).\end{aligned} \tag{2.38}$$

In terms of the functional $\Gamma(\Phi,\mathcal{B})$ the relation (2.38) reads

$$\Gamma(\Phi,\mathcal{B})\int dx\frac{\overleftarrow{\delta}}{\delta\mathcal{B}^{\alpha}_{\mu}}D^{\alpha\beta}_{\mu}(\mathcal{B})\omega_{\beta}$$
$$=-\Gamma(\Phi,\mathcal{B})\int dx\Big(\frac{\overleftarrow{\delta}}{\delta\mathcal{A}^{\alpha}_{\mu}}\mathcal{A}^{\gamma}_{\mu}+\frac{\overleftarrow{\delta}}{\delta\mathcal{C}^{\alpha}}\mathcal{C}^{\gamma}+\frac{\overleftarrow{\delta}}{\delta\overline{\mathcal{C}}^{\alpha}}\overline{\mathcal{C}}^{\gamma}+\frac{\overleftarrow{\delta}}{\delta\Phi^{\alpha}_{(B)}}\Phi^{\gamma}_{(B)}\Big)$$
$$\times gf^{\alpha\gamma\beta}\omega_{\beta}. \tag{2.39}$$

The relation (2.39) proves the invariance of $\Gamma(\Phi,\mathcal{B})$ under the gauge transformation of external vector field $\mathcal{B}$ accompanied by the tensor transformations of fields $\mathcal{A},\mathcal{C},\overline{\mathcal{C}},\Phi_{(B)}$,

$$\delta_\omega\mathcal{A}^{\alpha}_{\mu}=gf^{\alpha\gamma\beta}\mathcal{A}^{\gamma}_{\mu}\omega_{\beta},\ \delta_\omega\mathcal{C}^{\alpha}=gf^{\alpha\gamma\beta}\mathcal{C}^{\gamma}\omega_{\beta},$$
$$\delta_\omega\overline{\mathcal{C}}^{\alpha}=gf^{\alpha\gamma\beta}\overline{\mathcal{C}}^{\gamma}\omega_{\beta},\ \delta_\omega\Phi^{\alpha}_{(B)}=gf^{\alpha\gamma\beta}\Phi^{\gamma}_{(B)}\omega_{\beta}. \tag{2.40}$$

From (2.39) it follows the main property of functional $\Gamma(\mathcal{B})=\Gamma(\Phi,\mathcal{B})|_{\Phi=0}$ in the background field formalism[3]

$$\Gamma(\mathcal{B})\int dx\frac{\overleftarrow{\delta}}{\delta\mathcal{B}^{\alpha}_{\mu}}D^{\alpha\beta}_{\mu}(\mathcal{B})\omega_{\beta}=0. \tag{2.41}$$

The relations between the standard generating functionals and the analogous quantities in the background field formalism are established with modification of gauge functions likes to $\chi_\alpha(A,\mathcal{B})\to\chi'_\alpha(A,\mathcal{B})=\chi_\alpha(A,\mathcal{B})-\partial_\mu\mathcal{B}^{\alpha}_{\mu}$ [9].

3. Gauge invariance of average effective action

In this section we discuss the gauge invariance of average effective action for the FRG [26,27] in the background field formalism. Of course this issue is not new (see, for example, [33,34]), but we are going to demonstrate that requirement of gauge invariance of the average effective action restricts a tensor structure of regulator functions being essential objects of the approach. One of main ideas of the functional renormalization group approach was to modify behaviour of propagators of vector and ghost fields in IR and UV regions with the help of addition of a scale-dependent regulator action being quadratic in the fields. The scale-dependent regulator action

$$S_k(\phi)=\int dx\Big[\frac{1}{2}A^{\alpha}_{\mu}(x)R^{(1)\mu\nu}_{k\,\alpha\beta}(x)A^{\beta}_{\nu}(x)+\overline{C}^{\alpha}(x)R^{(2)}_{k\,\alpha\beta}(x)C^{\beta}(x)\Big] \tag{3.1}$$

is defined by regulator functions $R^{(1)\mu\nu}_{k\,\alpha\beta}(x), R^{(2)}_{k\,\alpha\beta}(x)$ which are independent of fields. The regulator functions $R^{(1)\mu\nu}_{k\,\alpha\beta}$ obey evident symmetry properties

$$R^{(1)\mu\nu}_{k\,\alpha\beta}=R^{(1)\nu\mu}_{k\,\beta\alpha}. \tag{3.2}$$

Let us require the invariance of $S_k(\phi)$ under transformations (2.21)

$$\delta_\omega S_k(\phi)=0. \tag{3.3}$$

It leads to the equations

$$f^{\alpha\beta\sigma}R^{(1)\mu\nu}_{k\,\sigma\gamma}+R^{(1)\mu\nu}_{k\,\alpha\sigma}f^{\sigma\gamma\beta}=0,\quad f^{\alpha\beta\sigma}R^{(2)}_{k\,\sigma\gamma}+R^{(2)}_{k\,\alpha\sigma}f^{\sigma\gamma\beta}=0, \tag{3.4}$$

[3] In the present paper we do not discuss a role of the BRST- and background gauge symmetries and problems connected with renormalization program for gauge theories within the background field method refereeing to the papers [18–20].

which can be presented in terms of Lie group generators $(t^{\alpha})_{\beta\gamma}=f^{\beta\alpha\gamma}$ as

$$[t^{\beta},R^{(1)\mu\nu}_{k}]_{\alpha\gamma}=0,\quad [t^{\beta},R^{(2)}_{k}]_{\alpha\gamma}=0. \tag{3.5}$$

Due to the Schur's lemma it follows from (3.5) that

$$R^{(1)\mu\nu}_{k\,\alpha\beta}=\delta_{\alpha\beta}R^{(1)\mu\nu}_{k},\quad R^{(2)}_{k\,\alpha\beta}=\delta_{\alpha\beta}R^{(2)}_{k}. \tag{3.6}$$

Therefore the regulator action (3.1) should be of the form

$$S_k(\phi)=\int dx\Big[\frac{1}{2}A^{\alpha}_{\mu}(x)R^{(1)\mu\nu}_{k}(x)A^{\alpha}_{\nu}(x)+\overline{C}^{\alpha}(x)R^{(2)}_{k}(x)C^{\alpha}(x)\Big] \tag{3.7}$$

to retain the invariance (3.3). In this case the full action

$$S_k(\phi,\mathcal{B})=S_{FP}(\phi,\mathcal{B})+S_k(\phi), \tag{3.8}$$

is invariant under transformations (2.21),

$$\delta_\omega S_k(\phi,\mathcal{B})=0. \tag{3.9}$$

The invariance (3.9) allows to extend all previous result concerning the gauge invariance problem on quantum level. The generating functionals of Green functions $Z_k(J,\mathcal{B})$ and connected Green functions $W_k(J,\mathcal{B})$ are defined by the functional integral

$$Z_k(J,\mathcal{B})=\int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[S_{FP}(\phi,\mathcal{B})+S_k(\phi)+J\phi\big]\Big\}$$
$$=\exp\Big\{\frac{i}{\hbar}W_k(J,\mathcal{B})\Big\}. \tag{3.10}$$

Repeating the same arguments as in previous section, we can proof the gauge invariance of the vacuum functional $Z_k(\mathcal{B})=Z_k(0,\mathcal{B})$ for the FRG approach in the background field formalism

$$\delta_\omega Z_k(\mathcal{B})=0,\quad \delta_\omega\mathcal{B}^{\alpha}_{\mu}=D^{\alpha\beta}_{\mu}(\mathcal{B})\omega_{\beta}. \tag{3.11}$$

From (3.9) and (3.10) it follows the gauge invariance of functional $W_k(\mathcal{B})=W_k(0,\mathcal{B})$ as well,

$$\delta_\omega W_k(\mathcal{B})=0. \tag{3.12}$$

In similar way we can proof the gauge invariance of average effective action $\Gamma_k(\Phi,\mathcal{B})=W_k(J,\mathcal{B})-J\Phi$,

$$\Gamma_k(\Phi,\mathcal{B})\frac{\overleftarrow{\delta}}{\delta\mathcal{B}^{\alpha}_{\mu}}D^{\alpha\beta}_{\mu}(\mathcal{B})\omega_{\beta}$$
$$=-\Gamma_k(\Phi,\mathcal{B})\Big(\frac{\overleftarrow{\delta}}{\delta\mathcal{A}^{\alpha}_{\mu}}\mathcal{A}^{\gamma}_{\mu}+\frac{\overleftarrow{\delta}}{\delta\mathcal{C}^{\alpha}}\mathcal{C}^{\gamma}+\frac{\overleftarrow{\delta}}{\delta\overline{\mathcal{C}}^{\alpha}}\overline{\mathcal{C}}^{\gamma}+\frac{\overleftarrow{\delta}}{\delta\Phi^{\alpha}_{(B)}}\Phi^{\gamma}_{(B)}\Big)$$
$$\times gf^{\alpha\gamma\beta}\omega_{\beta} \tag{3.13}$$

because the derivation of (3.13) operates in fact with the invariance of full action, $\delta_\omega(S_{FP}(\phi,\mathcal{B})+S_k(\phi))=0$, only. In particular, it follows from (2.12) the statement

$$\Gamma_k(\mathcal{B})\frac{\overleftarrow{\delta}}{\delta\mathcal{B}^{\alpha}_{\mu}}D^{\alpha\beta}_{\mu}(\mathcal{B})\omega_{\beta}=0,\quad \Gamma_k(\mathcal{B})=\Gamma_k(\Phi,\mathcal{B})|_{\Phi=0}, \tag{3.14}$$

concerning the invariance of $\Gamma_k(\mathcal{B})$ under the gauge transformations of external vector field.

4. Gauge dependence of average effective action

In this section we are going to investigate the gauge dependence problem for the FRG approach in the background field formalism. Standard formulation of this method being applied to gauge theories leads to ill defined the average effective action and the corresponding flow equation which still remain gauge dependent even on-shell [35,36]. The last feature of the FRG approach does not give a possibility of physical interpretations of results obtained.

To support our understanding the independence of gauge invariance and gauge dependence problems within background field formalism let us consider the generating functionals of Green functions and connected Green functions supplied with label "Ψ"

$$Z_{k\Psi}(J,\mathcal{B}) = \int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[S_{YM}(\mathcal{A}+\mathcal{B}) + \Psi(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B}) + S_k(\phi) + J\phi\big]\Big\} = \int d\phi\ \exp\Big\{\frac{i}{\hbar}S_k(\phi,\mathcal{B})\Big\} = \exp\Big\{\frac{i}{\hbar}W_{k\Psi}(J,\mathcal{B})\Big\}. \quad (4.1)$$

Taking into account that the regulator action does not depend on gauge we consider the functional (4.1) at $J=0$ corresponding another choice of the gauge fixing functional $\Psi \to \Psi + \delta\Psi$

$$Z_{k\Psi+\delta\Psi}(\mathcal{B}) = \int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[S_k(\phi,\mathcal{B}) + \delta\Psi(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B})\big]\Big\} = \exp\Big\{\frac{i}{\hbar}W_{k\Psi+\delta\Psi}(\mathcal{B})\Big\}, \quad (4.2)$$

where

$$\delta\Psi = \delta\Psi(\phi,\mathcal{B}) = \int dx \overline{C}^{\alpha}\delta\chi_\alpha(A,\mathcal{B}). \quad (4.3)$$

Now we are trying to compensate additional term $\delta\Psi\hat{R}$ in the exponent (4.2) using the changes of variables in the functional integral related closely to the symmetry of actions $S_{FP}(\phi,\mathcal{B})$ (2.14) and $S_k(\phi,\mathcal{B})$ (3.8). In the functional integral (4.2) we make first a change of variables in the form of the BRST transformations (2.9), (2.10), but trading the constant parameter μ to a functional $\Lambda = \Lambda(\phi,\mathcal{B})$. The action S_{FP} (2.12) is invariant under such change of variables but the action $S_k(\phi)$ (3.7) is not invariant, with the following variation

$$\delta S_k(\phi) = \int dx\Big(A^{\alpha}_{\mu}R^{(1)\mu\nu}D^{\alpha\beta}_{\nu}(A+\mathcal{B})C^{\beta} + \frac{1}{2}\overline{C}^{\alpha}R^{(2)}_k f^{\alpha\beta\gamma}C^{\beta}C^{\gamma} - B^{\alpha}R^{(2)}_k C^{\alpha}\Big)\Lambda. \quad (4.4)$$

The corresponding Jacobian J_1 reads

$$J_1 = \exp\Big\{-\int dx\Big(\frac{\delta\Lambda}{\delta A^{\alpha}_{\mu}}D^{\alpha\beta}_{\mu}(A+\mathcal{B})C^{\beta} + \frac{1}{2}f^{\alpha\beta\gamma}C^{\beta}C^{\gamma}\frac{\delta\Lambda}{\delta C^{\alpha}} + \frac{\delta\Lambda}{\delta\overline{C}^{\alpha}}B^{\alpha}\Big)\Big\}. \quad (4.5)$$

We make additionally a change of variables related to gauge transformations (2.23), (2.24) but using instead of parameters $\omega_\alpha(x)$ functions $\Omega_\alpha(x) = \Omega_\alpha(x,\phi(x),\mathcal{B}(x))$. The action $S_k(\phi,\mathcal{B})$ is invariant under these transformations but the relevant Jacobian, J_2 is not trivial,

$$J_2 = \exp\Big\{gf^{\alpha\beta\gamma}\int dx\Big(A^{\beta}_{\mu}(x)\frac{\partial\Omega_\gamma(x)}{\partial A^{\alpha}_{\mu}(x)} - C^{\beta}(x)\frac{\partial\Omega_\gamma(x)}{\partial C^{\alpha}(x)} - \overline{C}^{\beta}(x)\frac{\partial\Omega_\gamma(x)}{\partial\overline{C}^{\alpha}(x)}\Big)\Big\}. \quad (4.6)$$

If the condition,

$$J_1J_2\exp\Big\{\frac{i}{\hbar}\int dx\big[\delta\Psi(\phi,\mathcal{B})\hat{R}(\phi,\mathcal{B}) + \delta S_k(\phi)\big]\Big\} = 1, \quad (4.7)$$

is satisfied then the functional $Z_{k\Psi}(\mathcal{B})$ does not depend on gauge fixing functional Ψ. Having in mind the ghost numbers and Grassmann parities of functional Λ and functions $\Omega_\alpha(x)$

$$\mathrm{gh}(\Lambda) = -1,\quad \mathrm{gh}(\Omega_\alpha(x)) = 0,\quad \varepsilon(\Lambda) = 1,\quad \varepsilon(\Omega_\alpha(x)) = 0, \quad (4.8)$$

we have the following presentation in the lower power of ghost fields,

$$\Lambda = \Lambda^{(1)} + \Lambda^{(3)},\quad \Omega_\alpha(x) = \Omega^{(0)}_\alpha(x) + \Omega^{(2)}_\alpha(x), \quad (4.9)$$

where

$$\Lambda^{(1)} = \int dx\overline{C}^{\alpha}(x)\lambda^{(1)}_\alpha(x,A(x),\mathcal{B}(x)), \quad (4.10)$$

$$\Lambda^{(3)} = \int dx\frac{1}{2}\overline{C}^{\alpha}(x)\overline{C}^{\beta}(x)\lambda^{(3)}_{\alpha\beta\gamma}(x,A(x),\mathcal{B}(x))C^{\gamma}(x), \quad (4.11)$$

$$\Omega^{(0)}_\alpha(x) = \Omega^{(0)}_\alpha(x,A(x),\mathcal{B}(x)), \quad (4.12)$$

$$\Omega^{(2)}_\alpha(x,A(x),\mathcal{B}(x)) = \overline{C}^{\beta}(x)\omega^{(2)}_{\alpha\beta\gamma}(x,A(x),\mathcal{B}(x))C^{\gamma}(x). \quad (4.13)$$

Vanishing terms in (4.7) which don't depend on ghost fields $C,\overline{C}$ and auxiliary field B leads to the condition

$$\Omega^{(0)}_\alpha(x,A(x),\mathcal{B}(x)) = 0. \quad (4.14)$$

Consider in the equation (4.7) terms linear in B then we obtain

$$\lambda^{(1)}_\alpha(x,A(x),\mathcal{B}(x)) = \frac{i}{\hbar}\delta\chi_\alpha(x,A(x),\mathcal{B}(x)). \quad (4.15)$$

In turn analyzing the structures $B\overline{C}C$ in (4.7) we find the expression for $\lambda^{(3)}_{\alpha\beta\gamma}$,

$$\lambda^{(3)}_{\alpha\beta\gamma}(x,A,\mathcal{B}) = R^{(2)}(x)\big(\delta_{\beta\gamma}\lambda^{(1)}_\alpha(A,\mathcal{B}) - \delta_{\alpha\gamma}\lambda^{(1)}_\beta(A,\mathcal{B})\big), \quad (4.16)$$

$$\lambda^{(1)}_\alpha(A,\mathcal{B}) = \int dx\lambda^{(1)}_\alpha(x,A(x),\mathcal{B}(x)). \quad (4.17)$$

Vanishing structures $\overline{C}C$ leads to algebraic equations for $\omega^{(2)}_{\alpha\beta\gamma}$,

$$f^{\gamma\alpha\sigma}\omega^{(2)}_{\sigma\beta\gamma}(x,A(x),\mathcal{B}(x)) + f^{\gamma\beta\sigma}\omega^{(2)}_{\sigma\gamma\alpha}(x,A(x),\mathcal{B}(x)) = \frac{i}{g\hbar}D^{\gamma\alpha}_{\nu}(A+\mathcal{B})\big(A^{\gamma}_{\mu}(x)R^{(1)\mu\nu}_k(x)\big)\lambda^{(1)}_\beta(A,\mathcal{B}). \quad (4.18)$$

Therefore, in the case (4.9)-(4.18) we can reduce to zero in (4.7) all terms of the lowest order in fields $C,\overline{C},B$. Unfortunately, in its turn the $\lambda^{(3)}_{\alpha\beta\gamma}$ (4.16) creates the non-local term of structure $B\overline{C}\,\overline{C}CC$ which cannot be eliminated in a proposed scheme. It is necessary to add for functional Λ and functions Ω_α new terms of higher orders in ghost fields up to infinity. This situation looks unsatisfactory in terms of conventional quantum field theory and we are forced to restrict ourself by the case when $\Omega_\alpha = 0$ and $\Lambda = \Lambda^{(1)}$. Then we have

$$Z_{k\Psi+\delta\Psi}(\mathcal{B}) = \int d\phi\ \exp\Big\{\frac{i}{\hbar}\big[S_k(\phi,\mathcal{B}) + \delta S_k(\phi)\big]\Big\},\qquad Z_{k\Psi}(\mathcal{B}) \neq Z_{k\Psi+\delta\Psi}(\mathcal{B}). \quad (4.19)$$

Vacuum functional in the FRG approach within the background field formalism remains gauge dependent similar to the standard formulation [35,36]. The same statement is valid for elements of

S-matrix due to the equivalence theorem [32]. There are no problems deriving a modified Ward identity which is a consequence of BRST invariance of action $S_{FP}(\phi, \mathcal{B})$ and identities which follow from gauge invariance of the action $S_k(\phi, \mathcal{B})$ as well as to study gauge dependence of average effective action on-shell. We omit all these issues of the FRG approach because they do not help to solve the gauge dependence problem of results which are obtained within this method.

5. Summary

In the present paper we have analyzed the problems of the gauge invariance and gauge dependence of the generating functionals of Green functions in the background field formalism. It should be stressed that the gauge invariance of background effective action is usually under intensive study because it is a very important property for real calculations of Feynman diagrams. In turn the gauge dependence problem remains not in a focus of studies within this formalism although by itself this problem plays a principal role in our understanding of the ability to give a consistent physical interpretation of quantum results for gauge theories. We have supported this point of view by analysing the FRG approach in the background field formalism. We have shown that although the gauge invariance can be achieved with restrictions on the tensor structure of regulator functions but the gauge dependence problem cannot be solved in the existing representation of the FRG approach for gauge theories. The reason for this is the existing choice of regulator action (3.7). Consistent quantization of gauge theories permits modifications of quantum action (S_{FP} in the case of Yang-Mills theories) with the BRST-invariant additions only [37]. The regulator action (3.7) is not BRST-invariant that caused the gauge dependence problem.

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