

Research Article

An Area Rescaling Ansatz and Black Hole Entropy from Loop Quantum Gravity

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Received 12 December 2018; Accepted 25 February 2019; Published 1 April 2019

Academic Editor: Elias C. Vagenas

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Considering the possibility of ‘renormalization’ of the gravitational constant on the horizon, leading to a dependence on the level of the associated Chern-Simons theory, a rescaled area spectrum is proposed for the nonrotating black hole horizon in loop quantum gravity. The statistical mechanical calculation leading to the entropy provides a unique choice of the rescaling function for which the Bekenstein-Hawking area law is yielded without the need to choose the Barbero-Immirzi parameter (γ). γ is determined, rather than being chosen, by studying the limit in which the ‘renormalized’ gravitational constant on the horizon asymptotically approaches the ‘bare’ value. The possible physical dynamics behind the ‘renormalization’ is discussed.

1. Introduction

Loop quantum gravity (LQG) provides a platform for the calculation of entropy for nonrotating (assumed henceforth) black holes from the first principles, albeit in the kinematic framework [1]. The main criticism of this approach has been the necessity to choose a particular value of the Barbero-Immirzi parameter (γ), which is a dimensionless constant that characterizes the family of inequivalent kinematic quantization sectors of LQG, to obtain the Bekenstein-Hawking area law (BHAL) [2]. If the derivation is correct, then it is expected that one should get the BHAL without having to choose γ . As it appears, the full knowledge of the dynamics of LQG, the horizon degrees of freedom and the correct semi-classical limit of the theory are required to achieve this goal [2], which, unfortunately, does not seem to become available in near future. Nonetheless, the kinematic framework holds the potential to give us the hints towards the correct physical elements that give rise to the black hole entropy, which in turn may lead the path towards understanding the underlying dynamics. Here, I shall point out that there is a possibility of the involvement of a ‘renormalization’ of the gravitational constant *on the horizon* and incorporation of this effect in the entropy calculation leads us to the BHAL from LQG *without having to choose γ* . Since I shall base my arguments

on analogy, all the words associated with renormalization will appear in quotes. I shall heuristically argue that the quantum field theoretic structure that effectively describes the horizon degrees of freedom suggests that there is a possibility for a *rescaled* area spectrum to be used for the black hole horizon in LQG due to the ‘renormalization’ of the gravitational constant on the horizon. Further, the calculation of entropy with this rescaled area spectrum provides us with the unique rescaling function that leads to the BHAL without having to choose γ . The value of γ is *determined*, irrespective of obtaining the BHAL, by studying how the ‘renormalized’ gravitational constant on the horizon should *asymptotically* approach its ‘bare’ value in a limit that can be viewed as the ‘fixed point’ of the ‘renormalization group flow’ on the horizon. The novelty of this, albeit heuristic, work lies in the fact that the value of γ is now determined by a physical consistency requirement rather than being chosen just to match a desired result. Further, it appears that a ‘screening’ effect possibly is the physical cause that underlies the ‘renormalization’ procedure.

2. Motivation

The entire procedure of the entropy calculation for black holes in LQG consists of the following steps:

(1) *Horizon field dynamics*: the effective quantum field dynamics on the horizon (of topology $S^2 \times R$) is governed by a quantum Chern-Simons (CS) theory on a punctured 2-sphere and these punctures act as point-like sources coupled to the CS field strength [1]. The Hilbert space of this theory provides the state space of the horizon degrees of freedom that give rise to the entropy [1, 3, 4].

(2) *Spectrum of the source*: consider any arbitrary geometric 2-surface that is topologically S^2 . The quantum area of such a surface, in LQG, is given by $8\pi\gamma G \sum_{l=1}^N \sqrt{j_l(j_l + 1)}$ (setting $\hbar = c = 1$) and any j can take values like $0, 1/2, 1, \dots, \infty$. j_1, j_2, \dots, j_N are the quantum numbers carried by the intersection points (punctures) of the spin network edges with that 2-surface, N being the total number of punctures [5]. This is the same area spectrum that is used during the entropy calculation for black holes, with a crucial modification due to the interplay between quantum geometry and the CS theory on the horizon, that j can take values like $1/2, 1, \dots, k/2$, where $k = A_c/4\pi\gamma G$, A_c being the classical area of the black hole [1]. Hence, the contribution from an individual puncture (point-like source of the CS theory on the horizon) is $8\pi\gamma G \sqrt{j(j + 1)}$ with $j \in \{1/2, 1, 3/2, \dots, k/2\}$.

(3) *Statistical mechanics*: having the estimate of the microstate count from the first step and the area spectrum of the black hole from LQG in the second step, the statistical mechanics is applied to calculate the entropy.

Now, let me focus on the second step. It implies, in principle, the quantum area of an arbitrary geometric 2-surface of topology S^2 can be infinite, irrespective of the classical area of the surface. So, it is expected on physical grounds that this should *not* be the case when the concerned 2-surface is that of a physical object and the value of j should acquire an upper cut-off provided by the underlying physics associated with the surface of the physical object. This is exactly what happens for the black hole horizon. The value of j acquires an upper bound $k/2$, where the k is the level of the CS theory associated with the horizon; i.e., the first step plays a crucial role. Therefore, the theory governing the physics associated with the horizon naturally provides this upper bound. This is a result which is already manifested from the LQG kinematics and the effective horizon theory. However, the lack of knowledge about the full dynamics of a quantum black hole in LQG leaves room for some physics, associated with the horizon degrees of freedom contributing to the entropy of a black hole, which may be missing in the kinematics. As I shall argue, the information that is already available from the kinematics (the first step), indeed, hints towards such a possibility.

The field equations on a black hole horizon are that of a CS theory coupled to sources:

$$F = \frac{1}{k} \cdot \Sigma \quad (1)$$

where F is the curvature of the CS gauge fields on the horizon and Σ are the sources from the bulk. In the quantum theory, the source Σ is nonzero only at the punctures. Effectively, the theory on the horizon is a quantum CS theory coupled to point-like sources on a 2-sphere. The spectrum associated

with a single source is calculated for an arbitrary 2-sphere where there is no coupling with any field strength and it is given by

$$a_j = 8\pi\gamma G \sqrt{j(j + 1)}. \quad (2)$$

So, these punctures on an arbitrary 2-sphere are like ‘free’ excitations and the spectrum in (2) can be regarded as ‘bare’ spectrum. However, these ‘free’ excitations get coupled to the CS field strength in case the 2-sphere is a cross-section of a black hole horizon.

Now, in quantum field theory (QFT), the physical parameters like mass, charge, etc. associated with free particles get renormalized due to their coupling with fields, consequently affecting the mode spectrum. Analogously, in the present scenario, there is a possibility of γG in (2) (since γ and G always appear as a product in the kinematics of LQG, one should consider the ‘renormalization’ of γG rather than G alone [6]), which can be viewed as the ‘mode spectrum’ for the sources [7], getting ‘renormalized’ due to the coupling with the CS field strength. This ‘renormalized’ γG , say \tilde{G} , should depend on k , which is the cut-off for the allowed values of $2j$ that appears naturally in the theory on the horizon resulting from its gauge invariance [1]. Since the physical process involved with this ‘renormalization’ is associated with the quantum theory *on the horizon*, this \tilde{G} can only affect the microscopic physics localized on the horizon.

Although this heuristic ‘renormalization’ argument is only at the level of an analogy made from a QFT viewpoint, the possibility of the scenario cannot be completely ruled out unless one gets to know the full dynamics of the theory.

3. Rescaled Area Spectrum: An Ansatz

As I have just argued, \tilde{G} (the ‘renormalized’ γG), which enters the area spectrum of the black hole horizon, can only depend on k and on the value of γ because there are no other quantities intrinsic to the theory on the horizon. If δG is the change in the value of γG , then

$$\tilde{G}(k, \gamma) = \gamma G + \delta G(k, \gamma) \quad (3)$$

Since k and γ are both dimensionless, simply on dimensional grounds, $\delta G \propto G$. Further, as $k \rightarrow \infty$, δG must tend to zero because the sources get more weakly coupled to the CS field strength and the ‘renormalized’ spectrum should approach towards the ‘bare’ spectrum (this argument will be discussed more elaborately later).

Based on these arguments I propose that the area contribution from a single puncture with quantum number j , for a black hole horizon in LQG, is given by

$$a_j = 8\pi\tilde{G}(k, \gamma) \sqrt{j(j + 1)} \quad (4)$$

where

$$\tilde{G}(k, \gamma) = \sigma(k, \gamma) \gamma G \quad (5)$$

is the ‘renormalized’ gravitational constant on the horizon and $\sigma(k, \gamma)$ has the following properties:

- (i) Since the ‘bare’ spectrum needs to be positive definite, one has $\gamma > 0$. It is required that $\sigma > 0$ so that the ‘renormalized’ spectrum is positive definite too.
- (ii) $\lim_{k \rightarrow \infty} \sigma(k, \gamma) = 1$, i.e., as the sources get more weakly coupled to the CS field strength, \bar{G} asymptotically approaches γG .

Hence, I consider a *rescaled* area spectrum for the black hole horizon in LQG. As I shall show, the statistical mechanical calculation provides a unique choice of the function σ that leads to the BHAL and satisfies property (i). Satisfaction of property (ii) by σ , which is a physical consistency requirement, will determine γ . It is crucial to note that property (i) and property (ii) are independent of each other.

4. Entropy

I shall consider here black holes with classical area $A_c (\gg \mathcal{O}(G))$. Quantum area of a cross-section of a black hole horizon, with spin configuration $\{s_j\}$:

$$A_q = 8\pi\gamma\sigma G \sum_{j=1/2}^{k/2} s_j \sqrt{j(j+1)} \quad (6)$$

where $k := A_c/4\pi\gamma G$ and $s_j =$ number of punctures with quantum number j . Since j ranges from $1/2$ to $k/2$, hence $k \geq 1 \implies \gamma \leq A_c/4\pi G$. Also, since k is positive definite, the definition of k suggests that $\gamma > 0$. So, the quantum theory of the horizon is valid for $0 < \gamma \leq A_c/4\pi G$. Now, I shall implement the method of most probable distribution [8, 9] to calculate the microcanonical entropy of the black hole in the area ensemble. One can find the calculation (but, with the ‘bare’ spectrum) in [10]. So, I shall provide the main steps and results here to avoid an unnecessary repeat.

The microstate count for a spin configuration $\{s_j\}$ for which $A_q = A_c \pm \mathcal{O}(G)$:

$$\Omega[\{s_j\}] = N! \prod_{j=1/2}^{k/2} \frac{(2j+1)^{s_j}}{s_j!} \quad (7)$$

where $N := \sum_{j=1/2}^{k/2} s_j$ and $\{s_j\}$ satisfies the following constraint (considering $A_c \gg \mathcal{O}(G)$):

$$C : \sum_{j=1/2}^{k/2} s_j \sqrt{j(j+1)} - \frac{A_c}{8\pi\gamma\sigma G} = 0. \quad (8)$$

Then one finds the most probable configuration (MPC) by solving the following equation:

$$\delta \log \Omega[\{s_j\}] - \lambda \delta C = 0 \quad (9)$$

where λ is the Lagrange multiplier. This yields the distribution for the MPC to be

$$s_j^* = N_0 (2j+1) e^{-\lambda \sqrt{j(j+1)}} \quad (10)$$

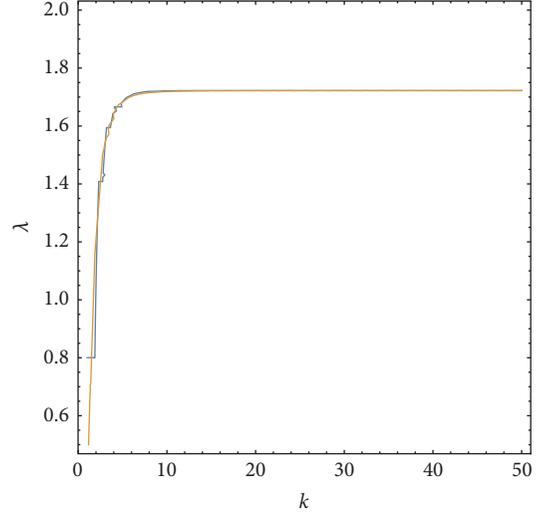


FIGURE 1: The blue curve shows the variation of λ as a function of k obtained from (12). The yellow curve shows the plot of the function $\lambda_0 \exp(-\alpha_0/(k+k_0)^{\nu_0})$ of k , with $\lambda_0 = 1.7220127$, $\alpha_0 = 27$, $k_0 = 1$, $\nu_0 = 4$.

where $N_0 := \sum_{j=1/2}^{k/2} s_j^*$ and the entropy comes out to be

$$S = \left(\frac{\lambda}{2\pi\gamma\sigma} \right) \cdot \frac{A_c}{4G}. \quad (11)$$

Now, a sum over j on both sides of (10) leads to the following consistency condition:

$$\sum_{j=1/2}^{k/2} (2j+1) e^{-\lambda \sqrt{j(j+1)}} = 1. \quad (12)$$

Equation (12), in principle, should lead to a solution for λ as a function of k . To avoid the mathematical complication of finding this solution analytically, I plot (using Mathematica) the function

$$\lambda = \lambda_0 \exp - \frac{\alpha_0}{(k+k_0)^{\nu_0}}, \quad (13)$$

where λ_0 , α_0 , k_0 , and ν_0 are some numbers. The plot (yellow coloured in Figure 1) fits to the curve obtained by plotting λ versus k from (12), up to a ‘very good’ approximation, for

$$\begin{aligned} \lambda_0 &= 1.7220127, \\ \alpha_0 &= 27, \\ k_0 &= 1, \\ \nu_0 &= 4. \end{aligned} \quad (14)$$

I do not provide here a mathematical estimate of ‘how good’ a fit it is. This is just an ‘optical’ fit obtained by numerical experiments.

It is manifested from the Figure 1 that the curves almost merge together for $k \geq 2$. Further, for black holes with $A_c \gg \mathcal{O}(G)$ one has $k \gg 1$ as, one will see shortly that γ is a number of order unity. Henceforth, I shall consider (13) as the functional dependence of λ on k .

4.1. *The BHAL.* From (11), it follows that the entropy is given by the BHAL, i.e.,

$$S = \frac{A_c}{4G}, \quad (15)$$

if the rescaling occurs as follows:

$$\sigma(k, \gamma) = \frac{\lambda(k)}{2\pi\gamma} = \frac{\lambda_0}{2\pi\gamma} \exp -\frac{\alpha_0}{(k + k_0)^{\nu_0}}. \quad (16)$$

Recalling that $k := A_c/4\pi\gamma G$, one can check that for all values of γ within the range $0 < \gamma \leq A_c/4\pi G$, σ is positive definite which is needed for the positive definiteness of the rescaled area spectrum. This was property (i) enlisted at the end of Section 3. Hence, I conclude that the statistical mechanical calculation with the horizon degrees of freedom in LQG leads to the BHAL *without having to choose* γ , if the area spectrum of the black hole horizon is rescaled by $\sigma(k, \gamma)$ given by (16).

However, σ needs to satisfy property (ii) as a requirement of physical consistency, as I shall explain in the next subsection. Importantly, I mention again and emphasize that property (ii) is independent of property (i).

4.2. *Determining γ from the ‘Fixed Point’.* All standard QFTs are some effective field theories valid until some energy scale. Only the renormalized quantities are calculable and measurable. The bare values of those physical quantities cannot be theoretically calculated. This is not unexpected because one does not have access to the most fundamental theory from which the corresponding QFT has come out to be an effective one. Taking quantum electrodynamics (QED) as an example, the bare electron charge is never measurable because one cannot decouple the electron from its field. However, if one *would have* known the most fundamental theory from which QED emerges effectively in some limit, then one could have expected to know, at least theoretically, the bare charge value of the electron. Added to this, the renormalized charge must have *asymptotically* approached that bare value in the high energy limit.

In the present scenario, one has the ‘bare’ area spectrum of an arbitrary 2-sphere and the rescaled (‘renormalized’) one on the black hole horizon. This is because one is now dealing with LQG which is one of the candidates of the fundamental theory of quantum gravity. Hence, the ‘bare’ quantities are expected to be known in this theory. Therefore, it seems quite logical to demand that $\lim_{k \rightarrow \infty} \sigma(k, \gamma) = 1$. To mention again, the physics underlying this limit is the following. The coupling strength $1/k$ of the point-like sources to the CS field strength decreases i.e., $1/k \rightarrow 0$. Hence, the area spectrum of the black hole should *asymptotically* approach the one of an arbitrary 2-sphere (the ‘bare’ spectrum) in this limit. In fact, one can view this limit as the ‘fixed point’ of the corresponding ‘renormalization group flow’ on the horizon, i.e., where the ‘beta function’ corresponding to the running gravitational constant on the horizon vanishes, namely,

$$\frac{d\bar{G}(k)}{d(\ln k)} = 0. \quad (17)$$

From (17), one can conclude that the ‘fixed point’ is implied by the limit $k \rightarrow \infty$.

One should be aware of the fact that this is an asymptotic limit and σ is never exactly unity since k is never exactly infinity. Putting $k = \infty$ in (1) will give $F = 0$ as the field equation on the horizon indicating that the sources have completely decoupled, which does not hold any meaning. This is analogous to the fact that if the bare charge of the electron were known, the renormalized charge would have only asymptotically approached that value in the high energy limit. However, it would have never exactly matched the bare value of the charge because that would have meant the electron has decoupled from its own field.

Now, using (5) and (16) it is trivial to check that in this limit, i.e., at the fixed point, \bar{G} asymptotically approaches γG only for a particular value of γ :

$$\begin{aligned} \bar{G}|_{\text{fixed point}} &= \gamma G \\ &\implies \lim_{k \rightarrow \infty} \sigma(k, \gamma) = 1 \\ &\implies \gamma = \frac{\lambda_0}{2\pi}. \end{aligned} \quad (18)$$

It may be noted that this is the exact value of γ that had to be *chosen* to obtain the BHAL in the usual practice [11]. However, the difference is that, as one can see, now γ is *determined* by a physical consistency requirement associated with the running gravitational constant on the horizon rather than being merely chosen to match a result.

Few comments: I shall make a digression here to offer some comments in relation to the available literature. It is very important to note that the present scenario is completely different from the one that was proposed in [2]. The renormalization of gravitational constant proposed in [2] is related to the renormalization of the fundamental degrees of freedom of LQG theory resulting in the general relativity emerging in the effective field theory limit. In this scenario there is a possibility that the gravitational constant can depend on the area of the black hole which creates the following problem (thanks to Daniel Sudarsky for pointing out this issue): what is the gravitational constant for a spacetime with more than one black hole? On the contrary, in the present scenario, I have proposed a ‘renormalization’ effect taking place only on the horizon due to the associated quantum theory. Since this effect is localized on the horizon, no problem arises in the presence of more than one black hole.

4.3. *Physical Dynamics of the ‘Renormalization’.* Taking into account (5), (16), and (18), I get

$$\bar{G}(k, \gamma) = \gamma G \exp -\frac{\alpha_0}{(k + k_0)^{\nu_0}} \quad (19)$$

with $\gamma = \lambda_0/2\pi$ and the values of λ_0 , α_0 , k_0 , and ν_0 are given in (14). Now, the obvious missing part to delve for is a reasonable explanation of the physical dynamics underlying the ‘renormalization’ procedure. The nature of dependence of \bar{G} on k , as given by (19), along with the kinematic structure of

the problem provides the stage for such an explanation which can be given as follows. One may note that for any finite k , $\bar{G} < \gamma G$. Therefore, the quantum gravitational flux carried by a single puncture with quantum number j for a black hole horizon of finite classical area (since $k := A_c/4\pi\gamma G$) is less than the same for an arbitrary geometric surface with equal classical area. Hence, there is a ‘screening’ effect that comes into play in the microscopic physics on the black hole horizon. However, one can ask the following question: what is special about a black hole horizon that makes it different from any arbitrary geometric surface? The answer goes as follows. A black hole horizon is characterized by a CS theory coupled to point-like sources and the Hilbert space of this theory provides the description of the microstates of the horizon, whereas an arbitrary geometric surface is devoid of such field dynamics. This is why a black hole horizon is a special surface. Given this feature of the horizon, there is an interesting consequence. As opposed to an arbitrary geometric surface, the quantum geometric excitations on the horizon are correlated in a specific way so that the microstates of the horizon satisfy the quantum CS equations [1]. While the kinematic aspect of this correlation leads to the microstate counting, there can also be a possible dynamical effect which can give rise to the ‘renormalization’ procedure under discussion. A possibility can be the following. The correlation of the quantum geometric excitations ‘all over’ the horizon causes an effective decrement of the gravitational flux carried by any individual puncture. This phenomenon can be viewed as a self-inflicted effective ‘screening’, caused by the mutual correlation of the punctures, on their own individual strengths, owing to their coupling with the CS gauge fields. Now, this effect should depend on the scale over which the correlation occurs, i.e., ‘all over’ the horizon. The scale which signifies ‘all over’ the horizon is the CS level k as it is a measure of the classical area of the horizon. This correlation, hence its dynamical effect - the ‘screening’, weakens with increasing k and decreasing coupling strength of the punctures with the CS gauge fields. Thus, the involved physics nicely falls into place with the fact that \bar{G} approaches γG as $k \rightarrow \infty$.

While the above explanation can be a possibility of the real physical dynamics behind the ‘renormalization’, one can pose the question that why the correlation causes ‘screening’ and not ‘antiscreening’. The answer to this question can only come from the study of the true dynamics which is hitherto unknown. Hence, the above discussed possibility cannot be ruled out yet.

5. Conclusion

Whatever I have discussed here is purely based on heuristic arguments that rely on some observations of the field theoretic structure that effectively describes the black hole horizon degrees of freedom in LQG and some analogies. This by no means is anything mathematically rigorous. However, having the knowledge of the full dynamics of quantum black holes in LQG yet out of reach, such a possibility of a ‘renormalized’ gravitational constant governing the microscopic physics on the horizon and giving rise to the

BHAL irrespective of the value of γ , cannot be ruled out completely. Also, I emphasize that the value of γ reported here has been *determined* by studying the asymptotic limit, of the ‘renormalized’ gravitational constant on the horizon, in which it approaches the ‘bare’ value. The limit can be viewed as the fixed point for the ‘renormalization group flow’ on the horizon, i.e., the beta function corresponding to the running gravitational constant on the horizon vanishes in this limit. Unlike the usual practice this is *not a choice* of γ to match the entropy with the BHAL. Added to this, the variation of the gravitational constant with the CS level on the horizon, which comes from the statistical mechanical calculation leading to the BHAL, indicates towards a ‘screening’ effect as an underlying physical cause behind the ‘renormalization’.

I hope this work may give a possible hint towards a more fundamental calculation of black hole entropy from LQG involving the underlying dynamics of quantum black hole horizons leading to the BHAL irrespective of the choice of γ .

Note that there is a belief in a part of the community that the BHAL has been derived from the LQG framework for arbitrary values of γ in [12]. However, in truth, there is a lower positive bound on γ in that derivation. The issue has been discussed with rigorous arguments in [13], which, unfortunately, has remained unnoticed.

Data Availability

The full article can be made available.

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

I am grateful to Benito Juarez Aubry and Tim Koslowski for carefully going through the manuscript and offering critical comments that have helped me in improving the same to a large extent. Most of the work was supported by DGAPA postdoctoral fellowship of UNAM, Mexico.

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