

Derivation of a gradient flow from the exact renormalization group

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We establish a concrete correspondence between a gradient flow and the renormalization group flow for a generic scalar field theory. We use the exact renormalization group formalism with a particular choice of the cutoff function.
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1. Introduction

The gradient flow, introduced in Refs. [1,2], has been attracting much attention lately. It is a continuous diffusion of local fields, well defined not only in continuum space but also on discrete lattices. The flow is much reminiscent of the renormalization group transformation [3], and it is especially important for lattice theories which had only discrete renormalization group transformations available.

The gradient flow has been used for scale setting and for the definition of a topological charge [2] (see Ref. [4] for a review). It has also been used to compute the expectation values of physical quantities such as the energy–momentum tensor via small diffusion time expansions [5] of local products of fields [6,7]. Conversely, the diffusion in the large-time limit leads to non-trivial infrared behaviors of theories such as scalar theories with $O(N)$ invariance and QCD with massless quarks; see, for instance, Refs. [8,9].

The similarity between the gradient flow and the renormalization group flow was already pointed out at the beginning [2] and has been pursued further [8–15]. The purpose of this paper is to establish a concrete correspondence between the two flows for a generic real scalar field theory in D -dimensional Euclidean space. We introduce Wilson actions with a finite momentum cutoff using the formalism of the exact renormalization group (ERG) [3]. Many readers may not have sufficient familiarity with the formalism, and we have chosen to give ample background material at the expense of the paper's brevity.

In the gradient flow we introduce a diffusion time $t > 0$, and extend the local field $\phi(x)$ in D -dimensional space by the solution of the diffusion equation (with no non-linear terms; see Ref. [16] for the motivation for this simple choice):

$$\partial_t \varphi(t, x) = \partial^2 \varphi(t, x), \quad (1)$$

where $\varphi(0, x) = \phi(x)$. We will show that the correlation functions of the diffused field $\varphi(t, x)$ evaluated in the bare action S_{bare} of the theory are given by the correlation functions of the elementary

field $\phi(x)$ evaluated in the Wilson action S_Λ with a finite momentum cutoff Λ :

$$(\varphi(t, x), S_{\text{bare}}) \iff (\phi(x), S_\Lambda), \quad (2)$$

where t and Λ are related by

$$t = \frac{1}{\Lambda^2}. \quad (3)$$

Based on the correspondence in Eq. (2), the renormalized nature of the diffused field results from the finite momentum cutoff of the Wilson action.

We organize the paper as follows. In Sect. 2 we give a brief overview of the ERG formalism; we provide more as we proceed. In Sect. 3 we derive a gradient flow from ERG. We consider a generic scalar theory, not necessarily renormalizable, and consider the behavior of the gradient flow at large diffusion times. In Sect. 4 we extend the gradient flow to renormalizable theories. We follow Sect. 12 of Ref. [3] to renormalize a theory non-perturbatively. This is to prepare for the discussion of the gradient flow at small diffusion times in Sect. 5, where we derive the small-time expansions of local products of the diffused field. In Sect. 6 we conclude the paper.

We use the following shorthand notation for the momentum integrals:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) \equiv (2\pi)^D \delta^{(D)}(p). \quad (4)$$

2. Overview of the ERG formalism

We give a brief overview of the exact renormalization group. There are many reviews available on the subject (see Ref. [17] and references therein); we follow the convention of Ref. [18] in the following.

Let $S_\Lambda[\phi]$ be a Wilson action of a real scalar field with a momentum cutoff Λ . The cutoff dependence of the Wilson action is determined so that the physics contents remain unchanged. We use the convention that the Boltzmann factor of functional integration is $e^{S_\Lambda[\phi]}$ rather than the more common $e^{-S_\Lambda[\phi]}$. The Λ dependence is given by the ERG differential equation in momentum space:

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} e^{S_\Lambda[\phi]} = & \int_p \left[\left(\frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \frac{\eta}{2} \right) \phi(p) \frac{\delta}{\delta \phi(p)} \right. \\ & \left. + \frac{\Delta(p/\Lambda) - \eta K(p/\Lambda) (1 - K(p/\Lambda))}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_\Lambda[\phi]}, \end{aligned} \quad (5)$$

where $K(p/\Lambda)$ is a positive cutoff function that decreases rapidly as $p \rightarrow \infty$, and

$$\Delta(p/\Lambda) \equiv \Lambda \frac{\partial}{\partial \Lambda} K(p/\Lambda). \quad (6)$$

We have introduced a constant anomalous dimension $\eta > 0$. Any cutoff function will do as long as

$$K(0) = 1, \quad \lim_{p \rightarrow \infty} K(p/\Lambda) = 0. \quad (7)$$

In this paper we choose

$$K(p/\Lambda) = e^{-p^2/\Lambda^2} \quad (8)$$

so that the inverse squared cutoff $1/\Lambda^2$ can play the role of a diffusion time t for the gradient flow.

Given a bare action $S_{\text{bare}}[\phi]$ at Λ_0 , we can solve Eq. (5) for $\Lambda < \Lambda_0$ by an integral formula [18]:

$$e^{S_\Lambda[\phi]} = \int [d\phi'] \exp \left[S_{\text{bare}}[\phi'] - \frac{1}{2} \int_p \frac{p^2}{\left(\frac{\Lambda}{\mu}\right)^\eta \frac{1-K(p/\Lambda)}{K(p/\Lambda)} - \left(\frac{\Lambda_0}{\mu}\right)^\eta \frac{1-K(p/\Lambda_0)}{K(p/\Lambda_0)}} \right. \\ \left. \times \left(\left(\frac{\Lambda_0}{\mu}\right)^{\frac{\eta}{2}} \frac{\phi'(p)}{K(p/\Lambda_0)} - \left(\frac{\Lambda}{\mu}\right)^{\frac{\eta}{2}} \frac{\phi(p)}{K(p/\Lambda)} \right) \left(\left(\frac{\Lambda_0}{\mu}\right)^{\frac{\eta}{2}} \frac{\phi'(-p)}{K(p/\Lambda_0)} - \left(\frac{\Lambda}{\mu}\right)^{\frac{\eta}{2}} \frac{\phi(-p)}{K(p/\Lambda)} \right) \right]. \quad (9)$$

The dependence on the reference momentum μ is only apparent. In the next section we use this formula to relate the correlation functions of S_Λ to the bare correlation functions.

As it is, the ERG differential equation of Eq. (5) has no fixed point. To obtain an ERG differential equation with a fixed point, we must measure dimensionful quantities in units of appropriate powers of the cutoff Λ . We introduce a dimensionless field with a dimensionless momentum by

$$\bar{p} \equiv \frac{p}{\Lambda}, \quad (10)$$

$$\bar{\phi}(\bar{p}) \equiv \Lambda^{\frac{D+2}{2}} \phi(p/\Lambda). \quad (11)$$

Defining $\tau \equiv \ln \frac{\mu}{\Lambda}$, we can rewrite Eq. (5) for

$$\bar{S}_\tau[\bar{\phi}] \equiv S_\Lambda[\phi] \quad (12)$$

as follows:

$$\partial_\tau e^{\bar{S}_\tau[\bar{\phi}]} = \int_{\bar{p}} \left[\left(\frac{\Delta(\bar{p})}{K(\bar{p})} + \frac{D+2}{2} - \frac{\eta}{2} + \bar{p} \cdot \partial_{\bar{p}} \right) \bar{\phi}(\bar{p}) \frac{\delta}{\delta \bar{\phi}(\bar{p})} \right. \\ \left. + \frac{\Delta(\bar{p}) - \eta K(\bar{p})(1-K(\bar{p}))}{\bar{p}^2} \frac{1}{2} \frac{\delta^2}{\delta \bar{\phi}(\bar{p}) \delta \bar{\phi}(-\bar{p})} \right] e^{\bar{S}_\tau[\bar{\phi}]} \quad (13)$$

With an appropriate choice of η , this can have a non-trivial fixed point action \bar{S}^* for which the right-hand side above vanishes.

3. Derivation of a gradient flow

To derive a gradient flow for the scalar field, we need to rewrite Eq. (9) for $\bar{S}_\tau[\bar{\phi}]$. The calculation is straightforward, and we just write down the result:

$$e^{\bar{S}_\tau[\bar{\phi}]} = \int [d\phi'] \exp \left[S_{\text{bare}}[\phi'] - \frac{1}{2} \int_{\bar{p}} \frac{\bar{p}^2}{\frac{1-K(\bar{p})}{K(\bar{p})} - \left(\frac{\Lambda_0}{\Lambda}\right)^\eta \frac{1-K(\bar{p} \frac{\Lambda}{\Lambda_0})}{K(\bar{p} \frac{\Lambda}{\Lambda_0})}} \right. \\ \left. \times \left(\left(\frac{\Lambda_0}{\Lambda}\right)^{\frac{\eta}{2}} \frac{\Lambda^{\frac{D+2}{2}} \phi'(\bar{p}\Lambda)}{K(\bar{p} \frac{\Lambda}{\Lambda_0})} - \frac{\bar{\phi}(\bar{p})}{K(\bar{p})} \right) \left(\left(\frac{\Lambda_0}{\Lambda}\right)^{\frac{\eta}{2}} \frac{\Lambda^{\frac{D+2}{2}} \phi'(-\bar{p}\Lambda)}{K(\bar{p} \frac{\Lambda}{\Lambda_0})} - \frac{\bar{\phi}(-\bar{p})}{K(\bar{p})} \right) \right]. \quad (14)$$

We introduce the generating functionals for \bar{S}_τ and S_{bare} :

$$\bar{Z}_\tau[\bar{J}] \equiv \int [d\bar{\phi}] \exp \left[\bar{S}_\tau[\bar{\phi}] + \int_{\bar{p}} \bar{J}(-\bar{p}) \bar{\phi}(\bar{p}) \right], \quad (15)$$

$$Z_0[J'] \equiv \int [d\phi'] \exp \left[S_{\text{bare}}[\phi'] + \int_p J'(-p) \phi'(p) \right]. \quad (16)$$

By substituting Eq. (14) into Eq. (15), and integrating over $\bar{\phi}$ first, we obtain [17]

$$\begin{aligned}\bar{Z}_\tau[\bar{J}] &= \int [d\phi'] \exp \left[S_{\text{bare}}[\phi'] + \int_{\bar{p}} \bar{J}(-\bar{p}) K(\bar{p}) \left(\frac{\Lambda_0}{\Lambda} \right)^{\frac{\eta}{2}} \frac{\Lambda^{\frac{D+2}{2}} \phi'(\bar{p}\Lambda)}{K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)} \right. \\ &\quad \left. + \frac{1}{2} \int_{\bar{p}} \bar{J}(\bar{p}) \bar{J}(-\bar{p}) \frac{K(\bar{p})^2}{\bar{p}^2} \left\{ \frac{1-K(\bar{p})}{K(\bar{p})} - \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \frac{1-K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)}{K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)} \right\} \right] \\ &= Z_0[J'] \exp \left[\frac{1}{2} \int_{\bar{p}} \bar{J}(\bar{p}) \bar{J}(-\bar{p}) \frac{K(\bar{p})^2}{\bar{p}^2} \left\{ \frac{1-K(\bar{p})}{K(\bar{p})} - \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \frac{1-K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)}{K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)} \right\} \right], \quad (17)\end{aligned}$$

where J' is given by

$$J'(\bar{p}\Lambda) \equiv \bar{J}(\bar{p}) \frac{K(\bar{p})}{K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)} \left(\frac{\Lambda_0}{\Lambda} \right)^{\frac{\eta}{2}} \Lambda^{-\frac{D-2}{2}}. \quad (18)$$

The extra quadratic term in the \bar{J} only affect the two-point function.

The result in Eq. (17) implies that the two-point function of \bar{S}_τ differs from that of S_{bare} by normalization and a shift, both momentum dependent:

$$\begin{aligned}\langle \bar{\phi}(\bar{p}) \bar{\phi}(\bar{q}) \rangle_{\bar{S}_\tau} &= \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \Lambda^{D+2} \left(\frac{K(\bar{p})}{K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)} \right)^2 \langle \phi(\bar{p}\Lambda) \phi(\bar{q}\Lambda) \rangle_{S_{\text{bare}}} \\ &\quad + \delta(\bar{p} + \bar{q}) \frac{K(\bar{p})^2}{\bar{p}^2} \left\{ \frac{1-K(\bar{p})}{K(\bar{p})} - \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \frac{1-K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)}{K\left(\bar{p}\frac{\Lambda}{\Lambda_0}\right)} \right\}. \quad (19)\end{aligned}$$

The connected parts of the higher-point functions are simply related by the same change of normalization as

$$\langle \bar{\phi}(\bar{p}_1) \cdots \bar{\phi}(\bar{p}_n) \rangle_{\bar{S}_\tau}^{\text{conn}} = \left(\left(\frac{\Lambda_0}{\Lambda} \right)^\eta \Lambda^{D+2} \right)^{\frac{\eta}{2}} \prod_{i=1}^n \frac{K(\bar{p}_i)}{K\left(\bar{p}_i\frac{\Lambda}{\Lambda_0}\right)} \cdot \langle \phi(\bar{p}_1\Lambda) \cdots \phi(\bar{p}_n\Lambda) \rangle_{S_{\text{bare}}}^{\text{conn}}, \quad (20)$$

where $n \neq 2$. (These results are well known in the ERG literature; see, for example, the review article in Ref. [17].)

To understand the above results better, let us introduce a Λ -dependent field by

$$\varphi(t, p) \equiv \frac{K(p/\Lambda)}{K(p/\Lambda_0)} \phi(p), \quad (21)$$

where the diffusion time t is given by

$$t \equiv \frac{1}{\Lambda^2} - \frac{1}{\Lambda_0^2} = \frac{e^{2\tau}}{\mu^2} - \frac{1}{\Lambda_0^2}. \quad (22)$$

Using the explicit form in Eq. (8) of the cutoff function, we obtain

$$\varphi(t, p) = e^{-tp^2} \phi(p). \quad (23)$$

In coordinate space, this gives

$$\varphi(t, x) \equiv \int_p e^{ipx} \varphi(t, p), \quad (24)$$

which satisfies the diffusion equation

$$(\partial_t - \partial^2) \varphi(t, x) = 0 \quad (25)$$

and the initial condition

$$\varphi(0, x) = \phi(x) \equiv \int_p e^{ipx} \phi(p). \quad (26)$$

Integrating Eqs. (19) and (20) over the momenta, and using the notation φ , we obtain

$$\langle \bar{\phi}(x)^2 \rangle_{\bar{S}_\tau} = \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \Lambda^{2-D} \langle \varphi(t, x)^2 \rangle_{S_{\text{bare}}} + \int_{\bar{p}} \frac{K(\bar{p})(1-K(\bar{p}))}{\bar{p}^2}, \quad (27)$$

$$\langle \bar{\phi}(x)^n \rangle_{\bar{S}_\tau}^{\text{conn}} = \left(\left(\frac{\Lambda_0}{\Lambda} \right)^\eta \Lambda^{2-D} \right)^{\frac{n}{2}} \langle \varphi(t, x)^n \rangle_{S_{\text{bare}}} \quad (28)$$

where, assuming $\Lambda \ll \Lambda_0$ and $\eta < 2$, we have set $K(\bar{p}\Lambda/\Lambda_0) = 1$ in evaluating the constant shift in the two-point function. The above gives an explicit relation between the expectation values of the diffused fields with the bare action and those of the elementary fields with the Wilson action.

Using Eqs. (27) and (28), we obtain, for example,

$$\frac{\langle \varphi(t)^4 \rangle_{S_{\text{bare}}}^{\text{conn}}}{\langle \varphi(t)^2 \rangle_{S_{\text{bare}}}^2} = \frac{\langle \varphi(t)^4 \rangle_{S_{\text{bare}}}}{\langle \varphi(t)^2 \rangle_{S_{\text{bare}}}^2} - 3 = \frac{\langle \bar{\phi}^4 \rangle_{\bar{S}_\tau}^{\text{conn}}}{\left(\langle \bar{\phi}^2 \rangle_{\bar{S}_\tau} - \int_{\bar{p}} \frac{K(\bar{p})(1-K(\bar{p}))}{\bar{p}^2} \right)^2}. \quad (29)$$

Similarly, we obtain

$$\frac{\frac{1}{\Lambda^2} \langle \partial_\mu \varphi(t, x) \partial_\mu \varphi(t, x) \rangle_{S_{\text{bare}}}}{\langle \varphi(t)^2 \rangle_{S_{\text{bare}}}} = \frac{\langle \partial_\mu \bar{\phi}(x) \partial_\mu \bar{\phi}(x) \rangle_{\bar{S}_\tau} - \int_{\bar{p}} K(\bar{p})(1-K(\bar{p}))}{\langle \bar{\phi}^2 \rangle_{\bar{S}_\tau} - \int_{\bar{p}} \frac{K(\bar{p})(1-K(\bar{p}))}{\bar{p}^2}}. \quad (30)$$

Suppose the bare theory S_{bare} is critical so that \bar{S}_τ approaches a fixed point as $\tau \rightarrow \infty$ (hence $t \rightarrow \infty$):

$$\lim_{\tau \rightarrow \infty} \bar{S}_\tau = \bar{S}^*. \quad (31)$$

We then obtain

$$\frac{\langle \varphi(t)^4 \rangle_{S_{\text{bare}}} - 3}{\langle \varphi(t)^2 \rangle_{S_{\text{bare}}}^2} \xrightarrow{t \rightarrow \infty} \frac{\langle \bar{\phi}^4 \rangle_{\bar{S}^*}^{\text{conn}}}{\left(\langle \bar{\phi}^2 \rangle_{\bar{S}^*} - \int_{\bar{p}} \frac{K(\bar{p})(1-K(\bar{p}))}{\bar{p}^2} \right)^2}, \quad (32)$$

$$\frac{\frac{1}{\Lambda^2} \langle \partial_\mu \varphi(t, x) \partial_\mu \varphi(t, x) \rangle_{S_{\text{bare}}}}{\langle \varphi(t)^2 \rangle_{S_{\text{bare}}}} \xrightarrow{t \rightarrow \infty} \frac{\langle \partial_\mu \bar{\phi}(x) \partial_\mu \bar{\phi}(x) \rangle_{\bar{S}^*} - \int_{\bar{p}} K(\bar{p})(1-K(\bar{p}))}{\langle \bar{\phi}^2 \rangle_{\bar{S}^*} - \int_{\bar{p}} \frac{K(\bar{p})(1-K(\bar{p}))}{\bar{p}^2}}. \quad (33)$$

The large- t behavior of the left-hand sides have been calculated explicitly in the large- N limit of the $O(N)$ linear sigma model in $D = 3$ [13, 14].

4. Gradient flow for a renormalizable theory

We next consider a bare action S_{bare} that corresponds to a renormalizable theory. To discuss renormalization non-perturbatively, we need to construct a renormalized trajectory \bar{S}_τ that can be traced back to the fixed point \bar{S}^* under the ERG flow:

$$\lim_{\tau \rightarrow -\infty} \bar{S}_\tau = \bar{S}^*. \quad (34)$$

Let us outline the construction of the renormalized trajectory, following Sect. 12 of Ref. [3].

Given a bare action $S_{\text{bare}}[\phi]$ with momentum cutoff Λ_0 , let

$$\bar{S}_{\text{bare}}[\bar{\phi}] \equiv S_{\text{bare}}[\phi] \quad (35)$$

be the corresponding action for the dimensionless field

$$\bar{\phi}(\bar{p}) = \Lambda_0^{\frac{D+2}{2}} \phi(\bar{p}\Lambda_0). \quad (36)$$

We can take the dimensionless squared mass \bar{m}^2 as the free parameter of the bare action $\bar{S}_{\text{bare}}(\bar{m}^2)[\bar{\phi}]$. We assume that the theory is critical at

$$\bar{m}^2 = \bar{m}_{\text{cr}}^2. \quad (37)$$

This means that the solution \bar{S}_τ of the ERG differential equation Eq. (13) with the initial condition

$$\bar{S}_{\tau=0} = \bar{S}_{\text{bare}}(\bar{m}_{\text{cr}}^2) \quad (38)$$

satisfies

$$\lim_{\tau \rightarrow \infty} \bar{S}_\tau = \bar{S}^*. \quad (39)$$

We assume that the fixed point \bar{S}^* has only one relevant direction with scaling dimension $y > 0$. (Otherwise, we need to tune more than \bar{m}^2 .) Let $\bar{S}_\tau(g, \Lambda_0/\mu)$ be the solution of Eq. (13) satisfying the initial condition

$$\bar{S}_{\tau=0}(g, \Lambda_0/\mu) = \bar{S}_{\text{bare}}(\bar{m}^2(g, \Lambda_0/\mu)), \quad (40)$$

where

$$\bar{m}^2(g, \Lambda_0/\mu) = \bar{m}_{\text{cr}}^2 + g \left(\frac{\mu}{\Lambda_0} \right)^y. \quad (41)$$

μ is an arbitrary reference momentum scale where the parameter g is defined. Note that $\bar{m}^2(g, \Lambda_0/\mu)$ satisfies

$$\left(yg \frac{\partial}{\partial g} + \Lambda_0 \frac{\partial}{\partial \Lambda_0} \right) \bar{m}^2(g, \Lambda_0/\mu) = 0. \quad (42)$$

This implies that

$$\left(yg \frac{\partial}{\partial g} + \Lambda_0 \frac{\partial}{\partial \Lambda_0} \right) \bar{S}_{\tau=0}(g, \Lambda_0/\mu) = 0. \quad (43)$$

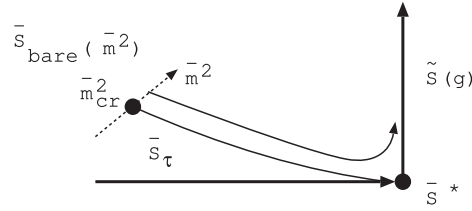


Fig. 1. ERG flows \bar{S}_τ and a renormalized trajectory $\tilde{S}(g)$.

We can then define a renormalized trajectory by the limit

$$\tilde{S}(g) \equiv \lim_{\Lambda_0 \rightarrow \infty} \bar{S}_{\ln \frac{\Lambda_0}{\mu}}(g, \Lambda_0/\mu). \quad (44)$$

For the limit to exist, we must find

$$\Lambda_0 \frac{\partial}{\partial \Lambda_0} \bar{S}_{\ln \frac{\Lambda_0}{\mu}}(g, \Lambda_0/\mu) \xrightarrow{\Lambda_0 \rightarrow \infty} 0. \quad (45)$$

For an explanation that such a limit exists, we refer the reader to standard references such as Sect. 12 of Ref. [3]. Since Eq. (43) implies that

$$\left(yg \frac{\partial}{\partial g} + \Lambda_0 \frac{\partial}{\partial \Lambda_0} \right) \bar{S}_\tau(g, \Lambda_0/\mu) = 0 \quad (46)$$

for any $\tau > 0$, we obtain, from Eq. (45),

$$yg \frac{\partial}{\partial g} \tilde{S}(g) = \lim_{\Lambda_0 \rightarrow \infty} \left(\Lambda_0 \frac{\partial}{\partial \Lambda_0} \bar{S}_{\ln \frac{\Lambda_0}{\mu}}(g, \Lambda/\mu) \right)_{\Lambda=\Lambda_0}. \quad (47)$$

Hence, from Eq. (13), $\tilde{S}(g)$ satisfies the ERG differential equation

$$yg \frac{\partial}{\partial g} e^{\tilde{S}(g)[\bar{\phi}]} = \int_p \left[\left(\frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \frac{\eta}{2} + p \cdot \partial_p \right) \bar{\phi}(p) \frac{\delta}{\delta \bar{\phi}(p)} + \frac{\Delta(p) - \eta K(p)(1 - K(p))}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} \right] e^{\tilde{S}(g)[\bar{\phi}]}, \quad (48)$$

where we have omitted the bar over the dimensionless momentum p to simplify the notation (see Fig. 1).

Now that we have constructed a renormalized trajectory $\tilde{S}(g)$, let us rewrite Eq. (44) as the relation between bare and renormalized Wilson actions of dimensionful fields. We first define a bare action with cutoff Λ_0 by

$$S_{\text{bare}}(g, \Lambda_0/\mu)[\phi] \equiv \bar{S}_{\text{bare}}(\bar{m}^2(g, \Lambda_0/\mu))[\bar{\phi}], \quad (49)$$

where

$$\phi(p) \equiv \Lambda_0^{-\frac{D+2}{2}} \bar{\phi}(p/\Lambda_0). \quad (50)$$

The squared mass of $S_{\text{bare}}(g, \Lambda_0/\mu)$ is given by

$$\Lambda_0^2 \bar{m}^2(g, \Lambda_0/\mu) = \Lambda_0^2 \bar{m}_{\text{cr}}^2 + g \Lambda_0^2 \left(\frac{\mu}{\Lambda_0} \right)^y. \quad (51)$$

We then define a renormalized Wilson action with cutoff Λ by

$$S_\Lambda[\phi] \equiv \tilde{S}(g_\Lambda)[\bar{\phi}], \quad (52)$$

where

$$\phi(p) \equiv \Lambda^{-\frac{D+2}{2}} \bar{\phi}(p/\Lambda) \quad (53)$$

and the dimensionless parameter g_Λ is defined by

$$g_\Lambda \equiv g \left(\frac{\mu}{\Lambda} \right)^y. \quad (54)$$

Since

$$\bar{m}^2(g_\Lambda, \Lambda_0/\Lambda) = \bar{m}^2(g, \Lambda_0/\mu), \quad (55)$$

Eq. (44) gives

$$\begin{aligned} \tilde{S}(g_\Lambda) &= \lim_{\Lambda_0 \rightarrow \infty} \bar{S}_{\ln \frac{\Lambda_0}{\mu}}(g_\Lambda, \Lambda_0/\mu) \\ &= \lim_{\Lambda_0 \rightarrow \infty} \bar{S}_{\ln \frac{\Lambda_0}{\Lambda}}(g_\Lambda, \Lambda_0/\Lambda) \\ &= \lim_{\Lambda_0 \rightarrow \infty} \bar{S}_{\ln \frac{\Lambda_0}{\Lambda}}(g, \Lambda_0/\mu). \end{aligned} \quad (56)$$

Since S_Λ is $\tilde{S}(g_\Lambda)$ for the dimensionful field of Eq. (53), and $S_{\text{bare}}(g, \Lambda_0/\mu)$ is $\bar{S}_{\text{bare}}(\bar{m}^2(g, \Lambda_0/\mu)) = \bar{S}_{\tau=0}(g, \Lambda_0/\mu)$ for Eq. (50), we find that S_Λ is obtained from the bare action $S_{\text{bare}}(g, \Lambda_0/\mu)$ by solving the ERG differential equation in Eq. (5) from Λ_0 to Λ (and taking the limit $\Lambda_0 \rightarrow \infty$). Hence, S_Λ and $S_{\text{bare}}(g, \Lambda_0/\mu)$ are related by Eq. (9) as

$$\begin{aligned} e^{S_\Lambda[\phi]} &= \int [d\phi'] \exp \left[S_{\text{bare}}(g, \Lambda_0/\mu)[\phi'] - \frac{1}{2} \int_p \frac{p^2}{\left(\frac{\Lambda_0}{\mu} \right)^\eta \frac{1-K(p/\Lambda)}{K(p/\Lambda)} - \left(\frac{\Lambda_0}{\mu} \right)^\eta \frac{1-K(p/\Lambda_0)}{K(p/\Lambda_0)}} \right. \\ &\quad \left. \times \left(\left(\frac{\Lambda_0}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi'(p)}{K(p/\Lambda_0)} - \left(\frac{\Lambda}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi(p)}{K(p/\Lambda)} \right) \left(\left(\frac{\Lambda_0}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi'(-p)}{K(p/\Lambda_0)} - \left(\frac{\Lambda}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi(-p)}{K(p/\Lambda)} \right) \right]. \end{aligned} \quad (57)$$

This implies that the correlation functions are related by

$$\begin{aligned} \langle \phi(p)\phi(q) \rangle_{S_\Lambda} &= \left(\frac{K(p/\Lambda)}{K(p/\Lambda_0)} \right)^2 \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \langle \phi(p)\phi(q) \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)} \\ &\quad + \delta(p+q) \frac{K(p/\Lambda)^2}{p^2} \left\{ \frac{1-K(p/\Lambda)}{K(p/\Lambda)} - \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \frac{1-K(p/\Lambda_0)}{K(p/\Lambda_0)} \right\} \\ &\quad \xrightarrow{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \langle \varphi(t,p)\varphi(t,q) \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)} \\ &\quad + \delta(p+q) \frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2}, \end{aligned} \quad (58)$$

and

$$\begin{aligned} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^{\text{conn}} &= \left(\frac{\Lambda_0}{\Lambda} \right)^{\frac{n}{2}\eta} \prod_{i=1}^n \frac{K(p_i/\Lambda)}{K(p_i/\Lambda_0)} \cdot \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)}^{\text{conn}} \\ &\xrightarrow{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\Lambda} \right)^{\frac{n}{2}\eta} \langle \varphi(t, p_1) \cdots \varphi(t, p_n) \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)}^{\text{conn}}, \end{aligned} \quad (59)$$

where the diffused field $\varphi(t, p)$ is given by Eqs. (22) and (23). Integrating over the momenta and taking $\Lambda_0 \rightarrow \infty$, we obtain the expectation values of local products:

$$\langle \phi(x)^2 \rangle_{S_\Lambda} = \lim_{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\Lambda} \right)^\eta \langle \varphi(t, x)^2 \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)} + \Lambda^{D-2} \int_p \frac{K(p)(1-K(p))}{p^2}, \quad (60)$$

$$\langle \phi(x)^n \rangle_{S_\Lambda}^{\text{conn}} = \lim_{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\Lambda} \right)^{\frac{n}{2}\eta} \langle \varphi(t, x)^n \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)}^{\text{conn}}. \quad (61)$$

Note that the diffused field only needs the standard wave function renormalization in the continuum limit $\Lambda_0 \rightarrow \infty$. Local products of $\phi(x)$ have no short-distance singularities thanks to the momentum cutoff Λ . Equations (60) and (61) give the concrete correspondence between the gradient flow (t) and RG flow (Λ) for the renormalized theory.

Before closing this section, we would like to relate the correlation functions in the continuum limit to those obtained by the Wilson action S_Λ . From Eqs. (58) and (59) we obtain

$$\lim_{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\mu} \right)^\eta \langle \phi(p)\phi(q) \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)} = \left(\frac{\Lambda}{\mu} \right)^\eta \langle \phi(p)\phi(q) \rangle_\Lambda, \quad (62)$$

$$\lim_{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\mu} \right)^{\frac{n}{2}\eta} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)}^{\text{conn}} = \left(\frac{\Lambda}{\mu} \right)^{\frac{n}{2}\eta} \langle \phi(p_1) \cdots \phi(p_n) \rangle_\Lambda^{\text{conn}}, \quad (63)$$

where we define

$$\langle \phi(p)\phi(q) \rangle_\Lambda \equiv \frac{1}{K(p/\Lambda)^2} \left(\langle \phi(p)\phi(q) \rangle_{S_\Lambda} - \frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2} \delta(p+q) \right), \quad (64)$$

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_\Lambda^{\text{conn}} \equiv \prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \cdot \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^{\text{conn}}. \quad (65)$$

The field of the Wilson action corresponds to a diffused field of the continuum limit, and we use the factor $1/K(p/\Lambda)$ for each $\phi(p)$ to reverse diffusion. Thus, using a Wilson action with a finite cutoff Λ , we manage to construct the correlation functions in the continuum limit, valid for any momenta.

The correlation functions with double brackets are the continuum limit defined at renormalization scale Λ . They satisfy the RG equation with anomalous dimension $\frac{\eta}{2}$:

$$\langle \phi(p)\phi(q) \rangle_{\Lambda'} = \left(\frac{\Lambda}{\Lambda'} \right)^\eta \langle \phi(p)\phi(q) \rangle_\Lambda, \quad (66)$$

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda'}^{\text{conn}} = \left(\frac{\Lambda}{\Lambda'} \right)^{\frac{n}{2}\eta} \langle \phi(p_1) \cdots \phi(p_n) \rangle_\Lambda^{\text{conn}}. \quad (67)$$

This explains the powers of Λ/μ , necessary to make the right-hand sides of Eqs. (62) and (63) independent of Λ .

5. The small-time expansions

In the previous section we obtained the relation in Eqs. (60) and (61) between the expectation value of $\varphi(t, x)^n$ in the continuum limit and that of $\phi(x)^n$ with the Wilson action S_Λ . We now wish to understand the behavior of the latter as $\Lambda \rightarrow \infty$, or equivalently $t \rightarrow 0$. In particular, we wish to derive small- t expansions analogous to those obtained for QCD in Ref. [5].

By construction, see Eqs. (52) and (53), we obtain

$$\langle \phi(p)\phi(q) \rangle_{S_\Lambda} = \Lambda^{-(D+2)} \langle \bar{\phi}(p/\Lambda)\bar{\phi}(q/\Lambda) \rangle_{\tilde{S}(g_\Lambda)}, \quad (68)$$

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^{\text{conn}} = \Lambda^{-(D+2)\frac{n}{2}} \langle \bar{\phi}(p_1/\Lambda) \cdots \bar{\phi}(p_n/\Lambda) \rangle_{\tilde{S}(g_\Lambda)}^{\text{conn}}. \quad (69)$$

Hence, integrating over the momenta, we obtain

$$\langle \phi^2 \rangle_{S_\Lambda} = \Lambda^{D-2} \langle \bar{\phi}^2 \rangle_{\tilde{S}(g_\Lambda)}, \quad (70)$$

$$\langle \phi^n \rangle_{S_\Lambda}^{\text{conn}} = \Lambda^{(D-2)\frac{n}{2}} \langle \bar{\phi}^n \rangle_{\tilde{S}(g_\Lambda)}^{\text{conn}}. \quad (71)$$

Let us introduce the dimensionless analogs of Eqs. (64) and (65) by

$$\langle \bar{\phi}(p)\bar{\phi}(q) \rangle_{g_\Lambda} \equiv \frac{1}{K(p)^2} \left(\langle \bar{\phi}(p)\bar{\phi}(q) \rangle_{\tilde{S}(g_\Lambda)} - \frac{K(p)(1-K(p))}{p^2} \delta(p+q) \right), \quad (72)$$

$$\langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{g_\Lambda}^{\text{conn}} \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{\tilde{S}(g_\Lambda)}^{\text{conn}}. \quad (73)$$

These satisfy the scaling laws

$$\langle \bar{\phi}(pe^\tau)\bar{\phi}(qe^\tau) \rangle_{g_\Lambda e^{\nu\tau}} = e^{-(D+2-\eta)\tau} \langle \bar{\phi}(p)\bar{\phi}(q) \rangle_{g_\Lambda}, \quad (74)$$

$$\langle \bar{\phi}(p_1 e^\tau) \cdots \bar{\phi}(p_n e^\tau) \rangle_{g_\Lambda e^{\nu\tau}}^{\text{conn}} = e^{-\frac{n}{2}(D+2-\eta)\tau} \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{g_\Lambda}^{\text{conn}}. \quad (75)$$

Correspondingly, the correlation functions in coordinate space, defined by

$$\langle \bar{\phi}(x)\bar{\phi}(0) \rangle_{g_\Lambda} \equiv \int_p e^{ipx} \int_q \langle \bar{\phi}(p)\bar{\phi}(q) \rangle_{g_\Lambda}, \quad (76)$$

$$\langle \bar{\phi}(x_1) \cdots \bar{\phi}(x_{n-1})\bar{\phi}(0) \rangle_{g_\Lambda}^{\text{conn}} \equiv \prod_{i=1}^{n-1} \int_{p_i} e^{ip_i x_i} \int_{p_n} \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{g_\Lambda}, \quad (77)$$

satisfy the scaling laws

$$\langle \bar{\phi}(xe^{-\tau})\bar{\phi}(0) \rangle_{g_\Lambda e^{\nu\tau}} = e^{(D-2+\eta)\tau} \langle \bar{\phi}(x)\bar{\phi}(0) \rangle_{g_\Lambda}, \quad (78)$$

$$\langle \bar{\phi}(x_1 e^{-\tau}) \cdots \bar{\phi}(x_n e^{-\tau}) \rangle_{g_\Lambda e^{\nu\tau}} = e^{\frac{n}{2}(D-2+\eta)\tau} \langle \bar{\phi}(x_1) \cdots \bar{\phi}(x_n) \rangle_{g_\Lambda}^{\text{conn}}. \quad (79)$$

Thus, we obtain

$$\begin{aligned} \langle \bar{\phi}^2 \rangle_{\tilde{S}(g_\Lambda)} &= \int_{p,q} \langle \bar{\phi}(p)\bar{\phi}(q) \rangle_{\tilde{S}(g_\Lambda)} \\ &= \int_{p,q} K(p)^2 \langle \bar{\phi}(p)\bar{\phi}(q) \rangle_{g_\Lambda} + \int_p \frac{K(p)(1-K(p))}{p^2} \\ &= \int d^D x \mathcal{K}_2(x) \langle \bar{\phi}(x)\bar{\phi}(0) \rangle_{g_\Lambda} + \int_p \frac{K(p)(1-K(p))}{p^2} \end{aligned} \quad (80)$$

and

$$\begin{aligned}
\langle \bar{\phi}^n \rangle_{\tilde{S}(g_\Lambda)}^{\text{conn}} &= \int_{p_1, \dots, p_n} \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{\tilde{S}(g_\Lambda)}^{\text{conn}} \\
&= \int_{p_1, \dots, p_n} \prod_{i=1}^n K(p_i) \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{g_\Lambda} \\
&= \int d^D x_1 \cdots d^D x_{n-1} \mathcal{K}_n(x_1, \dots, x_{n-1}) \langle \bar{\phi}(x_1) \cdots \bar{\phi}(x_{n-1}) \bar{\phi}(0) \rangle_{g_\Lambda}, \quad (81)
\end{aligned}$$

where we have defined

$$\mathcal{K}_2(x) \equiv \int_p e^{ipx} K(p)^2 = \frac{1}{(2\sqrt{2\pi})^D} \exp\left(-\frac{1}{8}x^2\right), \quad (82)$$

$$\begin{aligned}
\mathcal{K}_n(x_1, \dots, x_{n-1}) &\equiv \int_{p_1, \dots, p_{n-1}} e^{i \sum_{i=1}^{n-1} p_i x_i} \prod_{i=1}^{n-1} K(p_i) \cdot K(-p_1 - \cdots - p_{n-1}) \\
&= \frac{1}{(2^{n-1} \pi^{\frac{n-1}{2}} \sqrt{n})^D} \exp\left[-\frac{1}{4} \sum_{i=1}^{n-1} x_i^2 + \frac{1}{4n} (x_1 + \cdots + x_{n-1})^2\right]. \quad (83)
\end{aligned}$$

The functions \mathcal{K} are Gaussian with a range of order 1 in coordinate space.

Since the mass scale of $\tilde{S}(g_\Lambda)$ is of order $g_\Lambda^{\frac{1}{y}}$, the distance of order 1 is very short compared with the inverse mass $g_\Lambda^{-\frac{1}{y}}$ as long as $g_\Lambda \ll 1$. Because g_Λ is given by Eq. (54), we obtain $g_\Lambda \ll 1$ if we take

$$\Lambda \gg \mu g^{\frac{1}{y}}, \quad \text{or equivalently} \quad t = \frac{1}{\Lambda^2} \ll \frac{g^{-\frac{2}{y}}}{\mu^2}. \quad (84)$$

Hence, for such large Λ and the coordinates of order 1, we can use the short-distance expansions

$$\langle \bar{\phi}(x) \bar{\phi}(0) \rangle_{g_\Lambda} = \sum_i C_{2,i}(g_\Lambda; x) \langle \mathcal{O}_i(0) \rangle_{\tilde{S}(g_\Lambda)}, \quad (85)$$

$$\langle \bar{\phi}(x_1) \cdots \bar{\phi}(x_{n-1}) \bar{\phi}(0) \rangle_{g_\Lambda}^{\text{conn}} = \sum_i C_{n,i}(g_\Lambda; x_1, \dots, x_{n-1}) \langle \mathcal{O}_i(0) \rangle_{\tilde{S}(g_\Lambda)}, \quad (86)$$

where \mathcal{O}_i is a local composite operator of scale dimension $D - y_i$ whose expectation values are given by

$$\langle \mathcal{O}_i \rangle_{\tilde{S}(g_\Lambda)} = g_\Lambda^{\frac{D-y_i}{y}}. \quad (87)$$

The coefficient functions satisfy the RG equations:

$$(yg_\Lambda \partial_{g_\Lambda} - x \cdot \partial_x - (D - 2 + \eta - (D - y_i))) C_{2,i}(g_\Lambda; x) = 0, \quad (88)$$

$$\left(yg_\Lambda \partial_{g_\Lambda} - \sum_{i=1}^{n-1} x_i \cdot \partial_{x_i} - \left(\frac{n}{2}(D - 2 + \eta) - (D - y_i) \right) \right) C_{n,i}(g_\Lambda; x_1, \dots, x_{n-1}) = 0. \quad (89)$$

For x 's of order 1, the coefficient functions $C_{n,i}(g_\Lambda; x_1, \dots, x_{n-1})$ can be expanded in powers of $g_\Lambda \ll 1$. Hence,

$$C_{2,i}(g_\Lambda) \equiv \int d^D x \mathcal{K}_2(x) C_{2,i}(g_\Lambda; x), \quad (90)$$

$$C_{n,i}(g_\Lambda) \equiv \int d^D x_1 \cdots d^D x_{n-1} \mathcal{K}_n(x_1, \dots, x_{n-1}) C_{n,i}(g_\Lambda; x_1, \dots, x_{n-1}) \quad (91)$$

can be expanded in powers of g_Λ .

We thus obtain the large- Λ expansions as

$$\langle \bar{\phi}^2 \rangle_{\tilde{S}(g_\Lambda)} = \sum_i \langle \mathcal{O}_i \rangle_{\tilde{S}(g_\Lambda)} C_{2,i}(g_\Lambda), \quad (92)$$

$$\langle \bar{\phi}^n \rangle_{\tilde{S}(g_\Lambda)}^{\text{conn}} = \sum_i \langle \mathcal{O}_i \rangle_{\tilde{S}(g_\Lambda)} C_{n,i}(g_\Lambda). \quad (93)$$

Using Eqs. (60), (61), (70), and (71), we can rewrite the above for the continuum limit:

$$\lim_{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\mu} \right)^\eta \langle \varphi(t)^2 \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)} = \Lambda^{D-2} \left(\frac{\Lambda}{\mu} \right)^\eta \sum_i \langle \mathcal{O}_i \rangle_{\tilde{S}(g_\Lambda)} C_{2,i}(g_\Lambda), \quad (94)$$

$$\lim_{\Lambda_0 \rightarrow \infty} \left(\frac{\Lambda_0}{\mu} \right)^{n\frac{\eta}{2}} \langle \varphi(t)^n \rangle_{S_{\text{bare}}(g, \Lambda_0/\mu)}^{\text{conn}} = \Lambda^{(D-2)\frac{\eta}{2}} \left(\frac{\Lambda}{\mu} \right)^{n\frac{\eta}{2}} \sum_i \langle \mathcal{O}_i \rangle_{\tilde{S}(g_\Lambda)} C_{n,i}(g_\Lambda). \quad (95)$$

This is the analog of the small- t expansions obtained for QCD in Ref. [5]. Here, we have derived them by relating them to the standard short-distance expansions of Eqs. (85) and (86).

6. Conclusions

In this paper we have considered the gradient flow of a real scalar field obeying the simple diffusion equation without potential terms. We have then shown that the correlation functions of diffused fields match with those of elementary fields of a Wilson action that has a finite momentum cutoff. We have only discussed formalism, and we plan to provide concrete examples of the correspondence in a future publication.

Obviously we have scratched only the tip of an iceberg. In theories such as gauge theories and non-linear sigma models, the fields are continuous but live naturally in a compact space, and the diffusion equations that respect the geometry of the compact space should be and have been introduced [2,19]. Both gauge theories and non-linear sigma models can be formulated in ERG, but the realization of symmetry is not manifest (see Ref. [17], for example). The exact manner of the correspondence between the gradient flow and RG flow is not obvious, but we would be surprised if there were not any.

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