Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 943 (2019) 114614
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Information paradox in a Kerr-Newman black hole under generalized Hawking radiation

Xin-Yang Wang, Wen-Biao Liu *

Department of Physics, Beijing Normal University, Beijing 100875, China

Received 12 December 2018; received in revised form 1 April 2019; accepted 6 April 2019
Available online 10 April 2019
Editor: Clay Córdova

Abstract

Analogous to the calculation method of the interior volume inside a spherically symmetric black hole, we calculate the interior volume of a Kerr-Newman black hole. After that, we propose the generalized Stefan-Boltzmann law which can be used to investigate Hawking radiation with energy, charge and angular momentum. Based on it, the proportional relation between the entropy of the scalar field in the interior volume of the Kerr-Newman black hole and Bekenstein-Hawking entropy under generalized Hawking radiation has been investigated. Comparing to Hawking radiation without charge and angular momentum, we find that no matter the particles radiated from the black hole take charge and angular momentum or not, the two types of entropy remain the same proportional relation expression. Furthermore, the proportionality coefficient of the two types of entropy has been analyzed and discussed. It is found that the two types of entropy are approximately proportional to each other in an infinitesimal process except the late stage of Hawking radiation. Moreover, the proportional relation can degenerate to the Schwarzschild case when the charge and angular momentum of the black hole completely disappear. It illustrates that, for a Kerr-Newman black hole, different from Hawking radiation carrying only energy, the extremal black hole will technically not prevent and stop the evaporation anymore under generalized Hawking radiation, and the proportional relation obtained is applicable to investigate the evolution relation between the two types of entropy under an arbitrary Hawking radiation. Finally, based on the evolution of the proportional relation under generalized Hawking radiation, the black hole information paradox is brought up again and discussed more deeply.

* Corresponding author.
E-mail addresses: xinyang_wang@foxmail.com (X.-Y. Wang), wbliu@bnu.edu.cn (W.-B. Liu).

https://doi.org/10.1016/j.nuclphysb.2019.114614

1. Introduction

Christodoulou and Rovelli [1] have extended the volume definition inside a sphere from flat spacetime to curved spacetime, and then the interior volume of a spherically symmetric black hole can be defined. Based on the definition, the interior volume of a Schwarzschild black hole has been calculated, and it is found that the volume increases linearly with the advanced time. Subsequently, this definition has been extended from a spherically symmetric black hole to an axially symmetric black hole. After that, the interior volume of a Kerr and a Kerr-Newman black hole has been calculated, and the results are similar to the Schwarzschild case as increasing with advanced time too [2–4].

Besides, in order to solve the information paradox, the entropy of the quantum field modes in the interior volume of a Schwarzschild black hole has been calculated [5]. It is shown that the entropy increases linearly with the advanced time because it is proportional to the interior volume of the black hole. Moreover, the entropy is found to be always proportional to the Bekenstein-Hawking entropy during Hawking radiation. Subsequently, using this method, the entropy of quantum field modes in other kinds of black holes including a Kerr black hole has been calculated [3,6–8]. It is shown that the entropy increases linearly with the advanced time too, because it is proportional to the interior volume. Moreover, for all kinds of black holes, the proportional relation between the entropy and the Bekenstein-Hawking entropy under Hawking radiation is preserved approximately except the late stage of Hawking radiation.

However, in the previous works [3,5–8], all the particles radiated from the black hole are uncharged and without angular momentum in Hawking radiation. This treatment oversimplifies the evaporation process, and causes the extremal black hole to appear, and prevents the evaporation process in the end technically because of cosmic censorship conjecture. Fortunately, Hiscock and Weems [9] have investigated the Hawking radiation of a charged black hole and have derived the charge loss rate of the black hole during the evaporation process using the Schwinger formula, and then the modified Stefan-Boltzmann law is constructed qualitatively which can describe the energy and charge loss rate of a charged black hole under Hawking radiation. On the base of this literature, another modified Stefan-Boltzmann law [10] is constructed qualitatively, which can describe the energy and angular momentum loss rate of a Kerr black hole under Hawking radiation. By checking the two modified Stefan-Boltzmann laws, we find that the corrected terms are similar to the dJ and dQ terms in the first law of black hole thermodynamics. Hence, the modified Stefan-Boltzmann law can be seemingly regarded as consistent with the first law of thermodynamics. Following this idea, we propose the generalized Stefan-Boltzmann law to describe Hawking radiation with energy, charge and angular momentum, and then using this generalized law, we investigate the relation between the entropy of quantum field modes in the interior volume of a Kerr-Newman black hole and the Bekenstein-Hawking entropy under generalized Hawking radiation. Finally, we compare this relation with the case of Hawking radiation without charge and angular momentum.

The organization of the paper is as follows. In section 2, we review the interior volume of Schwarzschild, Kerr and Kerr-Newman black holes. In section 3, we construct qualitatively the generalized Stefan-Boltzmann law, and then the proportional relation between the entropy of quantum field modes in the interior volume of a Kerr-Newman black hole and the Bekenstein-Hawking entropy under generalized Hawking radiation is obtained. Thinking of Hawking radiation without charge and angular momentum, we can go back to the previous proportional relation again and compare the simplified results in detail with the case of generalized Hawking radiation. The paper ends with discussions and conclusions in section 4.

2. The interior volume of a Kerr-Newman black hole

The definition of the interior volume inside a spherically symmetric black hole has been proposed by Christodoulou and Rovelli [1]. This definition can be expressed as that the volume of the largest hypersurface inside the black hole bounded by two-sphere in the event horizon is just the interior volume of the black hole. Based on the definition, the interior volume of a Schwarzschild black hole which is formed by the collapse of a spherically symmetric object has been investigated. The numerical analysis shows that the largest hypersurface can be divided into three parts. The central part of the largest hypersurface is a long stretch at nearly constant radius $r = \frac{3}{2}M$, to which the event horizon $r = 2M$ attaches through the null hypersurface at one end, and the center of collapsing object $r = 0$ attaches at the other end. The volume of the null hypersurface is zero and the hypersurface in the collapsing object can contribute finite volume to the interior volume. However, the finite volume can be ignored because the central part of the largest hypersurface increases linearly with advanced time v. Therefore, the interior volume of a Schwarzschild black hole can be regarded as the volume of the spacelike hypersurface at $r = \frac{3}{2}M$ for the large advanced time v.

Subsequently, the definition of the interior volume is extended from a spherically symmetric black hole to an axially symmetric black hole, and the interior volume of a Kerr black hole has been investigated [2]. Due to that the Kerr metric is much more complicated than the Schwarzschild case, the analytical expression of the interior volume is more difficult to obtain using the method in Ref. [1]. Therefore, similar to the results in Ref. [1], the particular hypersurface bounded by two-sphere in the event horizon at large advanced time is directly chosen in Ref. [2] and it is also formed by three parts. The first part of this hypersurface is "close to null" just inside the sphere, and joins the second part $r = constant$ hypersurface all the way down to the third part which is in the matter filled region. The hypersurface is closed up at the center of the collapsing object $r = 0$. The volume of the first part is zero because it is null hypersurface, and the volume of the third part can be ignored at large v because this volume is finite. Therefore, the contribution to the interior volume of the black hole mainly comes from the spacelike hypersurface at $r = constant$. Based on this idea, in Ref. [2], the volume expression of an arbitrary hypersurface at $r = constant$ is given, and the interior volume of the Kerr black hole is regarded as the maximal value of the volume expression when r takes the special value r_s. Subsequently, in Refs. [2–4], it is demonstrated that this hypersurface at $r = r_s$ is very close to the largest hypersurface in a Kerr black hole and the spinning of spacetime has weak influence on this hypersurface. Therefore, the volume of the hypersurface at $r = r_s$ can be approximately regarded as the interior volume of a Kerr black hole.

For a Kerr-Newman black hole, similar to the Kerr case, the analytic expression of interior volume is still difficult to obtain. Therefore, we also directly select a particular hypersurface inside the Kerr-Newman black hole using the method in Ref. [2]. This selected hypersurface can also be divided into three parts. The central part of the hypersurface is a long stretch at nearly constant radius $r = r_s$, to which the event horizon attaches through the null hypersurface at one end, while the center of collapsing object $r = 0$ attaches to it at the other end. In Ref. [4], it is demonstrated that the hypersurface at $r = r_s$ is also very close to the largest hypersurface in a Kerr-Newman black hole, then the volume of the hypersurface at $r = r_s$ can also be regarded as the interior volume of a Kerr-Newman black hole approximately. In the following, we calculate the volume expression of an arbitrary hypersurface at $r = constant$ in a Kerr-Newman black hole and find out the special value r_s which is corresponding to the maximal value of the volume expression.

The metric of a Kerr-Newman black hole in the Eddington-Finkelstein coordinates is [11]

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\rho^2} dv^2 + 2dv\,dr + \rho^2 d\theta^2 + \frac{A \sin^2\theta}{\rho^2} d\phi^2 \\ &- 2a \sin^2\theta\, dr\, d\phi - \frac{2(2Mr - Q^2)}{\rho^2} a \sin^2\theta\, dv\, d\phi, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \rho^2 &\equiv r^2 + a^2 \cos^2\theta, \\ \Delta &\equiv r^2 - 2Mr + a^2 + Q^2, \\ A &\equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta. \end{aligned} \tag{2}$$

In which, $a = \frac{J}{M}$ is the angular momentum of unit mass of the black hole. From the Kerr-Newman metric, the volume expression of an arbitrary hypersurface at $r = constant$ inside the black hole can be obtained as

$$V_\Sigma = 2\pi v f_{max}(M, a, Q), \tag{3}$$

where

$$f_{max}(M, a, Q) = \sqrt{2Mr - r^2 - a^2 - Q^2} \left(\sqrt{r^2 + a^2} + \frac{r^2}{2a} \ln \frac{\sqrt{r^2 + a^2} + a}{\sqrt{r^2 + a^2} - a} \right) \Bigg|_{r = r_s}. \tag{4}$$

It can be found that the volume expression has a maximum value at $r_s = 1.3005$ when $M = 1$, $a = 0.5$, $Q = 0.5$. Therefore, based on the above statement, Eq. (3) can be regarded as the interior volume expression of a Kerr-Newman black hole approximately when $r = r_s$.

3. Interior and exterior entropy variation under generalized Hawking radiation

Hawking [12] first proposed that the emitted radiation from a black hole is thermal and its detailed form is independent of the structure of matter that collapsed to form the black hole. In other words, the thermal radiation does not contain the information of matter. From the view of quantum theory, this thermal radiation process makes a pure quantum state, which represents the state of the matter entering into the black hole, to transform into the mixed state. This transformation would destroy the information of the original quantum state and it violates Liouville's theorem. Therefore, the evaporation process of a black hole presents information paradox. In 2000, Parikh and Wilczek [13] proposed Hawking radiation as the tunneling process and showed that the radiation spectrum is not precisely thermal. It means that the information of the black hole may hide in Hawking radiation and propagate to infinity. Recently, Corda [14,15] improved the tunneling picture and proposed the "Bohr-like" approach to solve the black hole information paradox. In this perspective, for an excited black hole, the frequencies of quasi-normal mode can be naturally interpreted in terms of quantum levels. It is similar to the energy levels of the electron in the semi-classical Bohr model of the structure of a hydrogen atom, so this black hole model is called "Bohr-like" model. After that, based on the same model, a time dependent Schrödinger equation is given, which is used to describe the system composed of Hawking radiation and quasi-normal modes. In this equation, the physical state and the corresponding wave function can be written in terms of a unitary evolution matrix rather than a density matrix. Then, the final state is a pure quantum state instead of a mixed one. It is shown that using "Bohr-like" approach, Hawking radiation can be considered as unitary and the black hole information has never been lost during the evaporation process. This result can solve information paradox to some extent.

However, from another perspective, since the entropy is very closely associated with the information, investigating the evolution of the black hole entropy under Hawking radiation may also be used to investigate and solve the information paradox. Eq. (3) indicates that the interior volume of a Kerr-Newman black hole is proportional to the advanced time v. It means that, analogous to a Schwarzschild black hole, the interior volume of a Kerr black hole can also increase with the advanced time. This particular character of the interior volume can influence the statistical properties of the quantum field modes in the volume. One quite important quantity for the distribution of the quantum field modes is entropy. Therefore, the entropy of the quantum filed modes in the interior volume will increase naturally. Besides, the black hole interior volume may be a candidate to resolve the information paradox. Because a black hole can have a large amount of volume at the late advanced time, it can be used to store the lost information during Hawking radiation. According to Refs. [16–19], the lost information of the black hole during Hawking radiation may also be revealed in Bekenstein-Hawking entropy. In other words, decreasing Bekenstein-Hawking entropy with Hawking radiation can be related to the lost information of the black hole. If we can find the relation between the entropy of quantum field in the interior volume and Bekenstein-Hawking entropy, the entropy evolution of the interior (volume) and the exterior (event horizon) of the black hole under Hawking radiation can be connected. From this perspective, it is crucial to investigate the relation between the entropy of the quantum fields in the interior volume of the black hole and Bekenstein-Hawking entropy. In the following discussion, we only consider the massless scalar field in the interior volume of the black hole.

According to Ref. [5], although the interior volume of a Schwarzschild black hole can increase linearly with the advanced time, the equilibrium statistical method can also be used to calculate the entropy of the scalar field in the dynamic background. It is mainly because the largest hypersurface which corresponds to the interior volume has some special properties. Based on above discussions about the interior volume of a Schwarzschild black hole, the largest hypersurfaces in the black hole accumulate on the hypersurface at $r = \frac{3}{2}M$ at late advanced time. Therefore, near the hypersurface at $r = \frac{3}{2}M$, the proper time between two adjacent largest hypersurfaces tends to zero as the advanced time v increases. It means that time evolution "freezes" on the largest hypersurfaces inside a Schwarzschild black hole. So there is no evolution between these two adjacent largest hypersurfaces. Calculating the statistical properties of the scalar field in the interior volume can be seen as on the approximate simultaneity hypersurface. Moreover, this simultaneity hypersurface just corresponds to the interior volume of a Schwarzschild black hole. Therefore, the equilibrium statistical method can be used on the largest hypersurface to calculate the statistical properties of the scalar field in the interior volume. In Ref. [5], the entropy of the scalar field in the interior volume of a Schwarzschild black hole can be given as

$$S_\Sigma = \frac{\pi^2}{45\beta^3} V_\Sigma, \tag{5}$$

where β is the inverse temperature and V_Σ is the interior volume of the black hole. Following the idea in Ref. [5], the scalar field entropy in a Kerr-Newman black hole can also be calculated using the same method. Since the interior volume of a Kerr-Newman black hole can still be regarded as the volume of the hypersurface at $r = r_s$ approximately. Hence, analogous to the Schwarzschild black hole, near the hypersurface at $r = r_s$, there is still no evolution between two adjacent largest hypersurfaces as the advanced time v increases. It means that the equilibrium statistical method can also be used to calculate the statistical properties of the scalar field in the interior volume of a Kerr-Newman black hole. According to Ref. [3], the entropy of the scalar field in the interior volume of a black hole has been generally demonstrated that it can always be expressed as Eq. (5) using the equilibrium statistical method. Therefore, we can directly use Eq. (5) to investigate the variation of the scalar filed entropy under Hawking radiation.

Substituting Eq. (3) into Eq. (5), the scalar field entropy in the interior volume of a Kerr-Newman black hole can be expressed as

$$S_{\Sigma} = \frac{2\pi^3 f_{max}(M, a, Q)}{45\beta^3} v. \tag{6}$$

It is shown that the scalar field entropy increases with the advanced time, which means that the particular character of the interior volume can influence the statistical properties of the quantum field modes directly.

Next, we consider generalized Hawking radiation which describes the particles radiated from a Kerr-Newman black hole with energy, charge and angular momentum. Based on it, we want to find the connection between the evolution of the scalar field entropy in the interior volume and the evolution of Bekenstein-Hawking entropy under generalized Hawking radiation. In Eddington-Finkelstein coordinate system, for the stationary observer at infinity, the event horizon can be labeled by v. At each value of v, the section of a two sphere in the event horizon can be found, and then the volume of the largest hypersurface bounded by this section is the interior volume of the black hole. However, the interior volume increases linearly with the advanced time v. For different value of v, the interior volume bounded by the section in the event horizon is different. Moreover, the parameters of a Kerr-Newman black hole as mass M, charge Q and angular momentum a can also be labeled by v. Considering Hawking radiation, the parameters of the black hole can be regarded as decreasing with the advanced time v. So, the variation directions of the interior volume and the parameters of the black hole with the advanced time v are opposite under Hawking radiation. In addition, according to the above statement, considering an infinitesimal increasing of v, the largest hypersurface inside the black hole does not evolve with the proper time, it only extends along the hypersurface at $r = r_s$. For any special value of v, the equilibrium statistical method can be used to calculate the entropy of the scalar field in the interior volume. However, considering Hawking radiation, the interior volume increases with the advanced time v, the equilibrium statistical method cannot be used in this dynamical process. Therefore, in order to construct the connection between the entropy of the scalar field in the interior volume and the Bekenstein-Hawking entropy under Hawking radiation, two essential assumptions, which are the black-body radiation and the quasi-static process, are proposed in Ref. [3] reasonably. These assumptions can be summarized as follows:

(a) The black-body radiation assumption. The Hawking radiation of a black hole can be seen as black-body radiation. The temperature of the black hole for the stationary observer at infinity is Hawking temperature.
(b) The quasi-static process assumption. For an infinitesimal increasing of v, the interior volume is extending adiabatically, and Hawking radiation should satisfy $\frac{dM}{dv} \ll 1$, $\frac{dQ}{dv} \ll 1$ and $\frac{da}{dv} \ll 1$. It means that the evaporation process is slow enough and the thermal equilibrium between the scalar field inside the black hole and the event horizon is preserved in this adiabatic process.

According to assumption (a), for the stationary observer at infinity, the temperature of the black hole under Hawking radiation can be regarded as Hawking temperature at some special value of v. And then based on assumption (b), when the interior volume is increasing adiabatically and Hawking radiation is slow enough, a thermal equilibrium between the scalar field in the interior volume and the event horizon in an infinitesimal evaporation process can be constructed. It means that the temperature of the scalar field in the interior volume of the black hole can be regarded as Hawking temperature in an infinitesimal evaporation process. However, assumption (b) is not suitable for the late stage of Hawking radiation, because Hawking radiation can no longer be considered as a very slow process anymore. Therefore, the temperature of the scalar field in the black hole cannot be regarded as Hawking temperature at the late stage of Hawking radiation. From above discussion, the temperature of the scalar field can be regarded as Hawking temperature except the late stage of Hawking radiation, and the inverse of temperature β can be expressed as

$$\beta = \frac{1}{T} = \frac{2\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}\right)}{\sqrt{M^2 - a^2 - Q^2}}. \tag{7}$$

Substituting Eq. (7) into Eq. (6), the expression of the scalar field entropy in the black hole at some time v can be obtained as

$$S_\Sigma = \frac{1}{180} X(M, a, Q) v, \tag{8}$$

where

$$X(M, a, Q) = \frac{f_{max}(M, a, Q)\left(M^2 - a^2 - Q^2\right)^{3/2}}{\left(2M\sqrt{M^2 - a^2 - Q^2} + 2M^2 - Q^2\right)^3}. \tag{9}$$

Next, we consider the evolution of the scalar field entropy under generalized Hawking radiation. When the two assumptions are satisfied, in an infinitesimal evaporation process, the temperature of the scalar field in the black hole can be regarded as Hawking temperature and the statistical properties of the scalar field can be investigated using the equilibrium statistical method. It means that we should consider an infinitesimal evaporation process to investigate the evolution of the scalar field entropy inside the black hole under Hawking radiation. Moreover, using the assumption (b), the parameters of the black hole like mass M, charge Q and angular momentum a can be regarded as constants in an infinitesimal evaporation process, and their values are equal to the values at the beginning of the infinitesimal process. It is because the quasi-static process assumption should satisfy the conditions $\frac{dM}{dv} \ll 1$, $\frac{dQ}{dv} \ll 1$ and $\frac{dJ}{dv} \ll 1$. Therefore, for an arbitrary infinitesimal evaporation process except the late stage of Hawking radiation, the variation of scalar field entropy in the black hole can be expressed as the differential form. According to Eq. (8), the differential form of the scalar field entropy in a Kerr-Newman black hole can be written as

$$dS_\Sigma = \frac{1}{180}\left[X(M, a, Q)\, dv + v\frac{\partial X}{\partial M} dM + v\frac{\partial X}{\partial Q} dQ + v\frac{\partial X}{\partial a} da\right]. \tag{10}$$

Subsequently, we want to construct the connection between the entropy of the scalar field and Bekenstein-Hawking entropy under generalized Hawking radiation. The Bekenstein-Hawking entropy is defined as [20–22]

$$S_{BH} = \frac{A}{4}.$$

Taking the differentiation of this equation, we have

$$dS_{BH} = -\frac{2\pi}{\sqrt{M^2-a^2-Q^2}}\Big[\Big(\sqrt{M^2-a^2-Q^2}+M\Big)Q\,dQ + \Big(Q^2-2M\sqrt{M^2-a^2-Q^2}-2M^2\Big)dM + a\,dJ\Big]. \tag{11}$$

According to assumption (a), the evaporation process of black hole can be regarded as the black-body radiation, so the radiation process need to satisfy the Stefan-Boltzmann law. However, the simple Stefan-Boltzmann law can only be used to investigate the particles carrying only energy radiated from the black hole. Considering generalized Hawking radiation, the Stefan-Boltzmann law should be modified. In the previous work, the modified Stefan-Boltzmann law has been deduced qualitatively to describe the particles radiated from the charged black hole with both energy and charge in Ref. [9]. The basic idea is that the electron-positron pairs can be created near the event horizon by the vacuum fluctuation. The created particles with the different sign charge from the black hole will fall into the event horizon. Conversely, the created particles with the same sign charge as the black hole will be electromagnetically repelled towards infinity. Then the potential energy of the created particles relative to infinity is $\frac{Qe}{r_+}$, where r_+ is the radius of the event horizon, Q is the charge of the black hole and e is the charge of the created particles. Moreover, the charge of the created particles e can be regarded as the black hole's losing charge. Hence, the potential energy of the created particles in unit time can be written as $\frac{Q}{r_+}\frac{dQ}{dv}$. Therefore, the modified Stefan-Boltzmann law can be expressed as

$$\frac{dM}{dv} = -\sigma T^4 A + \frac{Q}{r_+}\frac{dQ}{dv}. \tag{12}$$

Moreover, based on this literature, we have constructed qualitatively the modified Stefan-Boltzmann law to describe the particles radiated from a Kerr black hole with both energy and angular momentum [10]. According to Penrose process and Hawking radiation explanation about the vacuum fluctuation near the event horizon, the particles with negative angular momentum produced by vacuum fluctuation will fall into the black hole along the negative energy orbit, which can reduce the angular momentum of the black hole. In the meantime, due to the conservation of angular momentum, the particles with positive angular momentum will carry the reduced angular momentum of the black hole away from the ergosphere and travel to infinity. The stationary observer at infinity can regard this total process as that the black hole does work on the outgoing particles while its angular momentum decreases. So, the radiation power of angular momentum can be written as $\Omega \times \frac{dJ}{dv}$, where Ω is the angular velocity of the created particles and $\frac{dJ}{dv}$ is the loss rate of the angular momentum of black hole. Moreover, the radiation particles are produced by vacuum fluctuation near the event horizon, the angular velocity of the outgoing particles can be viewed as the angular velocity of the event horizon $\frac{a}{r_+^2+a^2}$. Therefore, the modified Stefan-Boltzmann law can be expressed as

$$\frac{dM}{dv} = -\sigma T^4 A + \frac{a}{r_+^2+a^2}\frac{dJ}{dv}. \tag{13}$$

From Eq. (12) and Eq. (13), we can see that the correction terms in the modified laws are similar to the dJ and dQ terms in the first law of black hole thermodynamics respectively. So the modified Stefan-Boltzmann laws can be seemingly regarded as consistent with the first law of black hole thermodynamics. Hence, following this idea, we can construct qualitatively the generalized

Stefan-Boltzmann law, which can be used to investigate Hawking radiation with energy, charge and angular momentum. The generalized Stefan-Boltzmann law can be written as

$$\frac{dM}{dv} = -\sigma T^4 A + \frac{Q r_+}{r_+^2 + a^2}\frac{dQ}{dv} + \frac{a}{r_+^2 + a^2}\frac{dJ}{dv}, \tag{14}$$

where σ is a positive constant related to the number of quantized matter fields coupling with gravity [23], A is the area of event horizon, and r_+ is the radius of event horizon. Substituting Hawking temperature T, the area of horizon A, and the radius of horizon r_+ into Eq. (14), we have

$$\begin{aligned} dv = & \frac{4\pi^3 \left(-2M\sqrt{M^2-a^2-Q^2} - 2M^2 + Q^2\right)^2}{\sigma \left(a^2 - M^2 + Q^2\right)^2} \left[\left(\sqrt{M^2-a^2-Q^2} + M\right) Q\, dQ \right. \\ & \left. + \left(Q^2 - 2M\sqrt{M^2-a^2-Q^2} - 2M^2\right) dM + a\, dJ\right]. \end{aligned} \tag{15}$$

Comparing Eq. (11) and Eq. (15), we can see that these two equations contain the same dM, dQ and dJ terms. Since the differential form of Bekenstein-Hawking entropy is essentially the first law of black hole thermodynamics, it proves that the generalized Stefan-Boltzmann law should be consistent with the first law of black hole thermodynamics. Therefore, combining Eq. (15) with Eq. (11), the differential relation between the Bekenstein-Hawking entropy and the advanced time can be given as

$$dS_{BH} = -\frac{\sigma}{2\pi^2} Y(M, a, Q)\, dv, \tag{16}$$

where

$$Y(M, a, Q) = \frac{\left(M^2 - a^2 - Q^2\right)^{3/2}}{\left(-2M\sqrt{M^2-a^2-Q^2} - 2M^2 + Q^2\right)^2}. \tag{17}$$

Finally, combining Eq. (10) and Eq. (16), the proportional relation between the variation of the scalar field entropy and the variation of Bekenstein-Hawking entropy in an infinitesimal evaporation process can be expressed as

$$\frac{dS_\Sigma}{dS_{BH}} = -\frac{\pi^2}{90\sigma}\left[\frac{X(M,a,Q)}{Y(M,a,Q)} + \frac{v}{Y}\frac{\partial X}{\partial M}\frac{dM}{dv} + \frac{v}{Y}\frac{\partial X}{\partial Q}\frac{dQ}{dv} + \frac{v}{Y}\frac{\partial X}{\partial a}\frac{da}{dv}\right]. \tag{18}$$

According to the assumption (b), Hawking radiation should satisfy $\frac{dM}{dv} \ll 1$, $\frac{dQ}{dv} \ll 1$ and $\frac{da}{dv} \ll 1$, the three terms containing the partial derivative in this equation can be ignored. Therefore, this equation can be written as

$$\frac{dS_\Sigma}{dS_{BH}} = -\frac{\pi^2}{90\sigma}\frac{X(M,a,Q)}{Y(M,a,Q)}. \tag{19}$$

Substituting Eq. (9) and Eq. (17) into Eq. (19), and writing the differential proportional relation as the derivative form, the evolution relation between the entropy of the scalar field and Bekenstein-Hawking entropy under generalized Hawking radiation can be expressed as

$$\dot{S}_\Sigma = -\frac{\pi^2}{90\sigma} F(M, a, Q)\dot{S}_{BH}, \tag{20}$$

where

$$F(M,a,Q)=\frac{f_{max}(M,a,Q)}{2M\sqrt{M^2-a^2-Q^2}+2M^2-Q^2}, \tag{21}$$

and the dot indicates derivative by advanced time v.

In the following, we will calculate the evolution relation between the two types of entropy under Hawking radiation carrying only energy, and compare this result with the evolution relation under generalized Hawking radiation. If we only consider the particles with energy radiated from the Kerr-Newman black hole, the mass loss rate of the black hole should satisfy the simple Stefan-Boltzmann law, which can be given as [24]

$$\frac{dM}{dv}=-\sigma T^4 A. \tag{22}$$

Repeating the above deduction, the proportional relation in derivative form between the evolution of the entropy of scalar field in the interior volume and the evolution of Bekenstein-Hawking entropy under Hawking radiation can be obtained as

$$\dot{S}_\Sigma=-\frac{\pi^2}{90\sigma}F(M,a,Q)\dot{S}_{BH}, \tag{23}$$

where

$$F(M,a,Q)=\frac{f_{max}(M,a,Q)}{2M\sqrt{M^2-a^2-Q^2}+2M^2-Q^2}. \tag{24}$$

Comparing Eq. (20) and Eq. (23), we can find that no matter the particles take charge and angular momentum from a Kerr-Newman black hole or not, the two types of entropy remain the same proportional relation in the derivative form. This occurs because the generalized Stefan-Boltzmann law contains the similar dM, dQ and dJ terms as the first law of black hole thermodynamics. Hence, whether the parameters Q and J are regarded as variables cannot affect the final proportionality coefficient between the two types of entropy.

4. Discussions and conclusions

Although Eq. (24) is same as Eq. (21), they reveal in the evolution of the proportional relation between the entropy of the scalar field in the interior volume of the Kerr-Newman black hole and the Bekenstein-Hawking entropy during two different evaporation processes. Eq. (24) means the proportional relation between the two types of entropy under Hawking radiation carrying only energy, and the independent variable of the function $F(M,a,Q)$ is only M while the parameters a and Q can be considered as constants. In this situation, as the value of M decreases and approaches to a and Q, the extremal black hole will appear in the evaporation process. According to the cosmic censorship conjecture, the singularity needs to be hidden from an observer at infinity behind the event horizon of a black hole. In other words, the extremal black hole cannot be achieved in a physical process under Hawking radiation. This conjecture leads Hawking radiation to be terminated when the extremal black hole appears. However, in Eq. (21), the parameters M, a and Q are all independent variables of the function $F(M,a,Q)$. In this general situation, the evaporation process cannot be terminated by the extremal black hole necessarily, and the black hole can evaporate more generally.

The proportionality coefficient $F(M,a,Q)$ is shown in Fig. 1. Fig. 1 (a) shows the relation between function $F(M,a,Q)$ and the parameters M and a when Q is fixed, and Fig. 1 (b)

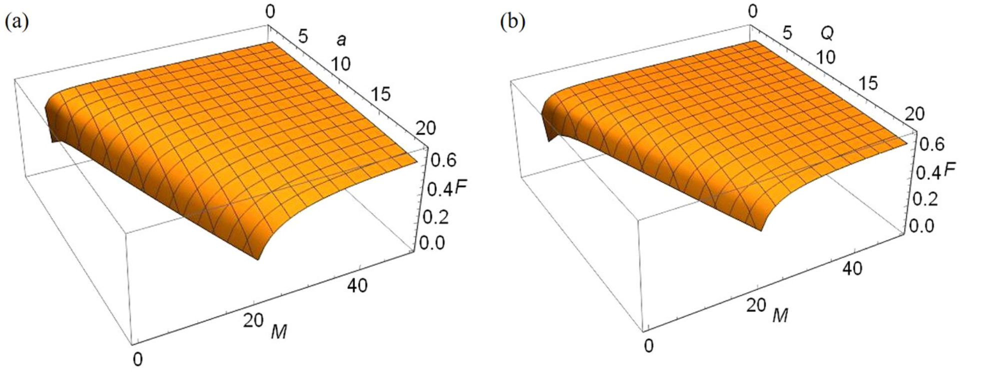

Fig. 1. (a) Plot of the parameters M and a versus $F(M, a, Q)$ when Q is fixed. (b) Plot of the parameters M and Q versus $F(M, a, Q)$ when a is fixed.

shows the relation between function $F(M, a, Q)$ and the parameters M and Q when a is fixed. The shape of $F(M, a, Q)$ in Fig. 1 (a) is almost the same as the one in Fig. 1 (b). The two figures show that at each specific value of a in Fig. 1 (a) or Q in Fig. 1 (b), the evolution of function $F(M, a, Q)$ can be divided into two stages. In the first stage, the trajectory of $F(M, a, Q)$ remains almost flat until the mass M drops to a "turning interval". It means that the evolution of the scalar field entropy in the interior volume is approximately proportional to the evolution of Bekenstein-Hawking entropy during the infinitesimal process in this stage. In the second stage, the trajectory of $F(M, a, Q)$ rapidly decreases, and it leads to the final stage of the extremal black hole, where the evaporation process terminates according to cosmic censorship conjecture. So, this evaporation process is reasonable and it will stop naturally. Besides, the length of flatness of the trajectory in the first stage increases as the charge or angular momentum of the black hole decreases. When the charge and angular momentum of the black hole disappear, the proportional relation in the Kerr-Newman black hole degenerates to the Schwarzschild case, while the two types of entropy are always proportional to each other during Hawking radiation. On the contrary, for generalized Hawking radiation, the Kerr-Newman black hole will evolve more freely and generally during the evaporation process. This result is different from the particles radiated from the black hole carrying only energy while the Hawking radiation will be terminated inevitably by the extremal black hole. However, according to Ref. [3], in the degenerating process as Schwarzschild case, the method of differential form can still not be applied in the final stage, because this stage cannot be regarded as a quasi-static process. Therefore, at the end of the black hole evaporation, using the differential form to obtain the proportional relation is meaningless.

In addition, the proportional relations between the variation of the scalar field entropy in a Kerr-Newman black hole and the variation of Bekenstein-Hawking entropy under Hawking radiation are given as Eq. (20) and Eq. (23). Although the form of proportional relation under generalized Hawking radiation is the same as that under Hawking radiation carrying only energy, they can give two different evaporation processes. For the situation of Hawking radiation carrying only energy, a Kerr-Newman black hole can only evolve toward an extremal black hole and the evaporation process can be terminated at the extremal case. In other words, Hawking radiation carrying only energy oversimplifies the evaporation process of a Kerr-Newman black hole. However, considering generalized Hawking radiation about the particles radiated from a Kerr-Newman black hole with energy, charge and angular momentum, the black hole evaporation can evolve more freely and generally. It means that, unlike Hawking radiation carrying only energy, an extremal black hole is not the unique final result of a Kerr-Newman black hole evolution. Actually, all the parameters of a Kerr-Newman black hole can decrease under generalized Hawking radiation. It leads to that the black hole can largely avoid the situation that the extremal black hole occurs, and it has a great probability to degenerate into a Schwarzschild black hole and disappear eventually. Therefore, in this situation, the extremal black hole can technically not prevent and stop the evaporation process, and it can evolve more freely and generally than the previous case. Therefore, the degenerate Hawking radiation carrying only energy is used to be compared with the generalized Hawking radiation. Moreover, it is also shown that the proportional relation obtained under generalized Hawking radiation is more general than that under Hawking radiation only with energy. It means that the proportional relation can be applied to investigate the evolution relation between the two types of entropy under an arbitrary evaporation process of a Kerr-Newman black hole.

Eq. (20) and Eq. (23) show that the entropy evolution of the interior and the exterior of the black hole is connected under two different Hawking radiation. The minus sign in them means that the entropy of the scalar field in the interior volume increases while the Bekenstein-Hawking entropy decreases under Hawking radiation. According to the above statement, the evolution of the scalar field entropy in a Kerr-Newman black hole is approximately proportional to the evolution of Bekenstein-Hawking entropy except the late stage of the evaporation process. Therefore, it can be considered that the evolution of the scalar field entropy in the interior volume has a particularly close connection to the evolution of Bekenstein-Hawking entropy under Hawking radiation. The lost information reflected by Bekenstein-Hawking entropy under Hawking radiation may be considered to be stored in the interior volume of the black hole. From this perspective, it is a possible way to resolve the information paradox of black hole. However, for Hawking radiation carrying only energy, the mass of the black hole will change with the evaporation process while the charge and angular momentum will keep constant. This treatment has major limitations to investigate the black hole information paradox. Considering generalized Hawking radiation, the black hole can evolve more generally, and we can investigate the general evolution behavior under Hawking radiation.

Till now, we have calculated the interior volume of a Kerr-Newman black hole. Thinking of the generalized Hawking radiation to describe the particles radiated from the black hole with energy, charge and angular momentum, the generalized Stefan-Boltzmann law has been proposed. Based on it, the proportional relation in derivative form between the entropy of the scalar field in the interior volume of the Kerr-Newman black hole and the Bekenstein-Hawking entropy can be given in an infinitesimal process. Subsequently, considering the particles radiated from the black hole with only energy, the proportional relation in derivative form is naturally obtained. The results show that whether the particles take charge and angular momentum from the black hole cannot influence the final proportional relation. Finally, the proportionality coefficient of the two types of entropy is analyzed. It is found that the evolution of the scalar field entropy in the interior volume is approximately proportional to the evolution of Bekenstein-Hawking entropy except the late stage of black hole evaporation. When the charge and angular momentum of the black hole become to zero, the proportional relation will degenerate to the Schwarzschild case. It means that, considering generalized Hawking radiation, the evaporation process of a Kerr-Newman black hole cannot necessarily to be terminated anymore by the extremal black hole. In our extended and general research here, the Kerr-Newman black hole can evolve more freely and generally. Moreover, comparing to Hawking radiation carrying only energy, the proportional relation obtained under generalized Hawking radiation, can be applied to investigate the evolution relation between the two types of entropy under an arbitrary evaporation process of a Kerr-Newman black hole. Furthermore, even considering generalized Hawking radiation, the evolution of the proportional relation between the two types of entropy can be regarded as a possible way to resolve the information paradox.

Acknowledgements

The authors thank Yi-Ru Wang for many useful discussions. This work is supported by the National Natural Science Foundation of China (Grant No. 11235003).

References

[1] M. Christodoulou, C. Rovelli, How big is a black hole?, Phys. Rev. D 91 (6) (2015) 064046, https://doi.org/10.1103/PhysRevD.91.064046, arXiv:1411.2854.
[2] I. Bengtsson, E. Jakobsson, Black holes: their large interiors, Mod. Phys. Lett. A 30 (21) (2015) 1550103, https://doi.org/10.1142/S0217732315501035, arXiv:1502.01907.
[3] X.-Y. Wang, J. Jiang, W.-B. Liu, Entropy in the interior of a Kerr black hole, Class. Quantum Gravity 35 (21) (2018) 215002, https://doi.org/10.1088/1361-6382/aae276, arXiv:1803.09649.
[4] X.-Y. Wang, W.-B. Liu, The interior volume calculation for an axially symmetric black hole, Phys. Lett. B 788 (2019) 464–467, https://doi.org/10.1016/j.physletb.2018.11.056, http://www.sciencedirect.com/science/article/pii/S0370269318309134.
[5] B. Zhang, Entropy in the interior of a black hole and thermodynamics, Phys. Rev. D 92 (8) (2015) 081501, https://doi.org/10.1103/PhysRevD.92.081501, arXiv:1510.02182.
[6] S.-Z. Han, J.-Z. Yang, X.-Y. Wang, W.-B. Liu, The entropy inside a charged black hole under hawking radiation, Int. J. Theor. Phys. 57 (11) (2018) 3429–3435, https://doi.org/10.1007/s10773-018-3856-6.
[7] J.-Z. Yang, W.-B. Liu, Entropy in the interior of a higher-dimensional black hole, Phys. Lett. B 782 (2018) 372–374, https://doi.org/10.1016/j.physletb.2018.05.050.
[8] S. Ali, X.-Y. Wang, W.-B. Liu, Entropy in a d-dimensional charged black hole, Int. J. Mod. Phys. A 33 (27) (2018) 1850159, https://doi.org/10.1142/S0217751X18501592.
[9] W.A. Hiscock, L.D. Weems, Evolution of charged evaporating black holes, Phys. Rev. D 41 (1990) 1142, https://doi.org/10.1103/PhysRevD.41.1142.
[10] X.-Y. Wang, W.-B. Liu, Hawking radiation with angular momentum and the entropy variation in a Kerr black hole, submitted for publication.
[11] S.Q. Wu, X. Cai, Hawking radiation of a nonstationary Kerr-Newman black hole: spin rotation coupling effect, Gen. Relativ. Gravit. 34 (2002) 605–617, https://doi.org/10.1023/A:1015929926316, arXiv:gr-qc/0111044.
[12] S.W. Hawking, Information loss in black holes, Phys. Rev. D 72 (2005) 084013, https://doi.org/10.1103/PhysRevD.72.084013, arXiv:hep-th/0507171.
[13] M.K. Parikh, F. Wilczek, Hawking radiation as tunneling, Phys. Rev. Lett. 85 (2000) 5042–5045, https://doi.org/10.1103/PhysRevLett.85.5042, arXiv:hep-th/9907001.
[14] C. Corda, Precise model of Hawking radiation from the tunnelling mechanism, Class. Quantum Gravity 32 (19) (2015) 195007, https://doi.org/10.1088/0264-9381/32/19/195007, arXiv:1411.1213.
[15] C. Corda, Time dependent Schrödinger equation for black hole evaporation: no information loss, Ann. Phys. 353 (2015) 71–82, https://doi.org/10.1016/j.aop.2014.11.002, arXiv:1304.1899.
[16] M. Christodoulou, T. De Lorenzo, Volume inside old black holes, Phys. Rev. D 94 (10) (2016) 104002, https://doi.org/10.1103/PhysRevD.94.104002, arXiv:1604.07222.
[17] D.N. Page, Information in black hole radiation, Phys. Rev. Lett. 71 (1993) 3743–3746, https://doi.org/10.1103/PhysRevLett.71.3743, arXiv:hep-th/9306083.
[18] D. Marolf, The black hole information problem: past, present, and future, Rep. Prog. Phys. 80 (9) (2017) 092001, https://doi.org/10.1088/1361-6633/aa77cc, arXiv:1703.02143.
[19] T. Jacobson, D. Marolf, C. Rovelli, Black hole entropy: inside or out?, Int. J. Theor. Phys. 44 (2005) 1807–1837, https://doi.org/10.1007/s10773-005-8896-z, arXiv:hep-th/0501103.
[20] S.W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199–220, https://doi.org/10.1007/BF02345020, Commun. Math. Phys. 167 (1975), https://doi.org/10.1007/BF01608497.

[21] J.D. Bekenstein, Black holes and the second law, Lett. Nuovo Cimento 4 (1972) 737–740, https://doi.org/10.1007/BF02757029.
[22] J.D. Bekenstein, Black holes and entropy, Phys. Rev. D 7 (1973) 2333–2346, https://doi.org/10.1103/PhysRevD.7.2333.
[23] L.E. Parker, D. Toms, Quantum Field Theory in Curved Spacetime: Quantized Field and Gravity, Cambridge University Press, 2009.
[24] I. Montvay, E. Pietarinen, Stefan-Boltzmann law at high temperature for the gluon gas, Phys. Lett. B 110 (1982) 148–154, https://doi.org/10.1016/0370-2693(82)91024-3.