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Superspace formulation with a new and extended BRST

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ABSTRACT

Superspace formulation is an elegant proposal to deal with BRST symmetries and Ward identities of a gauge theory. A new effective SU(N) QCD in a novel quadratic gauge has recently been introduced which throws light on interesting non perturbative characteristics of QCD. The effective theory has an unusual feature of having two pairs of ghosts and thus has some unconventional BRST transformations. In this paper, we propose a formulation of this unique theory in a six-dimensional superspace in which additional two dimensions are Grassmannian. The process of stated formulation also brings to attention a novel result that not all sets of BRST transformations under which the theory is invariant lead to consistent superspace version. The work is also imperative in view of its implications on the ward identity in the superspace.

1. Introduction

A symmetry in the theory gives rise to simple ward identities, with which the task of renormalizing the theory becomes simple and unambiguous. In a presence of the gauge condition, the effective gauge theory does not remain gauge invariant. Hence, there is a need to look for some other symmetry to make matters simple. The BRST symmetry which extends the usual gauge symmetry restores the lost invariance and thus helps in the renormalization program of the effective gauge theory. An anti commuting parameter in the BRST transformations suggests the construction of the superspace (or superfield) formulation which is an effective and geometrical proposal to make evident the underlying BRST and ward identity structure of a gauge theory in a simple way. The formulation begins with extending usual 4-dim space (x^μ, $\mu = 0, ..., 3$) to 6-dim superspace $X^i = (x^\mu, \theta, \lambda)$, $(i = 0, ..., 5)$ where additional two dimensions, θ and λ are anti commuting. The ward identity, for example, of Yang-Mills theory with a Lorenz gauge takes a simple superspace form of $\frac{\partial W_s}{\partial \theta} = 0$, where W_s is a superspace generating functional of the theory [1,2]. On the 6-dim superspace (X^i), the gauge supermultiplet $\tilde{\mathcal{A}}^a_i(x^\mu, \theta, \lambda) = \left(\tilde{A}^a_\mu(x^\mu, \theta, \lambda), F^a_4(x^\mu, \theta, \lambda), F^a_5(x^\mu, \theta, \lambda)\right)$ is constructed. A given superfield say, $\mathcal{B}$ can be expanded as

$$\mathcal{B}(x^\mu, \theta, \lambda) = B_1(x) + \theta B_2(x) + \lambda B_3(x) + \theta\lambda B_4(x). \quad (1)$$

There are mainly two approaches to the superspace formulation. The choice of Grassmannian superfields F_4, F_5 depends on the approach. In the one which we are going to follow and advocated in Refs. [1–4], both F_4 and F_5 will be identified with the ghost field only. Here an anti-ghost superfield is not the part of the gauge supermultiplet. This approach does not treat BRST and anti-BRST transformations on equal footing since coefficients of θ and λ both of a given superfield are identified with BRST transformations only. The general strategy of this approach is to establish equivalence of the 6-dim superspace Yang-Mills action to normal 4-dim one provided certain identifications with BRST transformations. We name this approach *the first approach*. We shall detail the first approach in the upcoming content. In the second approach [5–9], a superfield F_4 relates to a ghost field and F_5 relates to an anti-ghost field. The coefficient of θ is identified with the BRST transformation and λ is identified with the anti-BRST transformation. Therefore, the second method treats BRST and anti-BRST transformations on equal ground. The general strategy of this method which is reverse of the first approach is to derive BRST and anti-BRST transformations provided equivalence between superspace and normal 4-dim theory. Such equivalence is formally known as “Horizontality Condition (HC)” [5–12]. The inclusion of matter fields is straightforward in both approaches. In the second, so called gauge invariant restrictions (GIRs) are additionally put along with the HC [13] to include matter fields. Notations introduced so far will be consistently followed throughout the paper.

Recently a new quadratic gauge has been introduced in the SU(N) QCD in which interesting non perturbative characteristics of QCD has been brought to light [14–16]. The application of the

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quadratic gauge has also been extended to SO(N) QCD to investigate the non perturbative regime of the effective theory [17]. In the effective SU(N) theory, a fresh observation of identifying deconfinement to confinement phase transition as PT phase transition has been made and thus providing the first example of a gauge theory exhibiting the PT phase transition [18]. The gauge has several unusual features. (1) The gauge does not fall in the class of Abelian projection gauges and has quark confinement signatures. So far, studies of the confinement have been done in Abelian projection gauges. (2) It is the covariant algebraic gauge. In general, algebraic gauges are not covariant. (3) It removes the Gribov ambiguity on the compact manifold contrary to the case of usual gauges. [15].

In the present case, BRST transformations are readily known therefore the first approach is a natural choice to construct the consistent superspace formulation. It is a novel construction never tried before since the theory includes two pairs of ghosts whereas previous studies only included a pair of ghosts. The result is also imperative for the future objective of deriving Ward identity of the theory in the superspace.

The plan of the paper is now presented. In the next section, we review preliminaries of a superspace version with the first approach. In section 3, we briefly discuss the SU(N) QCD in the newly introduced quadratic gauge. In the section 4, we shall lay down the superspace formulation of the effective theory in the quadratic gauge. In the last section, the results of the paper will be summarized.

2. First approach of superspace formulation

We formulate the superspace version of the effective theory in Lorenz gauge with the first approach as a preliminary to the upcoming objective. One advantage this approach possesses for us is that the BRST symmetry of the theory is sufficient to obtain the equivalent theory in the superspace. The effective action in the Lorenz gauge is given as follows

$$\begin{aligned}\mathcal{S}_L &= \int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 - \overline{c^a}\partial^\mu (D_\mu c)^a\right)\\ &= \int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{\xi}{2}F^{a2} + F^a(\partial_\mu A^{\mu a}) - \overline{c^a}\partial^\mu (D_\mu c)^a\right),\end{aligned} \tag{2}$$

in the second version, the F^a are a set of auxiliary fields called Nakanishi-Lautrup fields, $(D_\mu c)^a = \partial_\mu c^a - gf^{abc}A^b_\mu c^c$ is a covariant derivative of the ghost c and the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - gf^{abc}A^b_\mu(x)A^c_\nu(x). \tag{3}$$

This action is invariant under following BRST transformations

$$\begin{aligned}\delta c^d &= \frac{\delta\omega}{2} f^{dbc}c^b c^c\\ \delta\overline{c^d} &= \frac{\delta\omega}{g}F^d\\ \delta A^d_\mu &= \frac{\delta\omega}{g}(D_\mu c)^d\\ \delta F^d &= 0.\end{aligned} \tag{4}$$

Now we proceed to extend this 4-dim theory in 6 dimensions, $X^i = (x^\mu, \theta, \lambda)$, $(i = 0, ..., 5)$ where $\theta^2 = \lambda^2 = 0$. Components of a generalized metric tensor, $\tilde{g}_{ij}$ are given to be $\tilde{g}_{ij} = g_{ij}$, when $0 \le i, j \le 3$, the g_{ij} is usual Minkowski metric of diagonal elements $(-1, 1, 1, 1)$; $-\tilde{g}_{45} = \tilde{g}_{54} = 1$; the rest of the $\tilde{g}^{ij}$ not mentioned here are zero. We recall the gauge supermultiplet to be $\tilde{\mathcal{A}}^a_i(x^\mu, \theta, \lambda) = \left(\tilde{A}^a_\mu(x^\mu, \theta, \lambda), F^a_4(x^\mu, \theta, \lambda), F^a_5(x^\mu, \theta, \lambda)\right)$ and demand that it transforms like a covariant vector under coordinate transformations in the superspace that preserve $\tilde{g}_{ij}X^iX^j$. These transformations are elements of Supersymplectic group OSp(3, 1|2). We define the gauge superfield $\tilde{A}^a_\mu$ as

$$\tilde{A}^a_\mu = A^a_\mu(x) + \theta\, A^a_{\mu,\theta}(x) + \lambda\, A^a_{\mu,\lambda}(x) + \theta\lambda\, A^a_{\mu,\theta\lambda}(x). \tag{5}$$

The Grassmann superfields F^a_4, F^a_5 are now generically defined as follows

$$F^a_k = c^a_k(x) + \theta\, c^a_{k\theta}(x) + \lambda\, c^a_{k\lambda}(x) + \theta\lambda\, c^a_{k\theta\lambda}(x). \tag{6}$$

Since we are dealing with the first approach, a component field c^a_k will be identified with ghost field c^a. With all these in mind, we generalize the field strength in Eq. (3) and an infinitesimal gauge transformation respectively in the superspace as follows

$$\tilde{\mathcal{F}}^a_{ij} = \partial_i\tilde{\mathcal{A}}^a_j - \tilde{\mathcal{A}}^a_i\overleftarrow{\partial_j} - gf^{abc}\tilde{\mathcal{A}}^b_i\tilde{\mathcal{A}}^c_j, \tag{7}$$

$$\tilde{\mathcal{A}'}^a_i = \tilde{\mathcal{A}}^a_i + (\partial_i\tilde{w}^a - gf^{abc}\tilde{\mathcal{A}}^b_i\tilde{w}^c), \tag{8}$$

where $\tilde{w}^a$ is defined on the superspace X^i i.e., $\tilde{w}^a(X)$; and as is evident from notations given above $\tilde{\mathcal{A}}_4 = F_4$, $\partial_4 = \partial_\theta$, $\tilde{\mathcal{A}}_5 = F_5$, $\partial_5 = \partial_\lambda$. We note that the right derivative $\overleftarrow{\partial}$ in the second term of RHS in Eq. (7) is put in order to render the generalization valid in two additional Grassmann dimensions. We observe that c-number field strength components are anti symmetric and grassmannian field strength components are symmetric i.e.,

$$\tilde{F}^a_{ij} = -\tilde{F}^a_{ji},\ 0 \le i \le 3 \textit{ and/or } 0 \le j \le 3 \tag{9}$$

$$\tilde{F}^a_{ij} = \tilde{F}^a_{ij},\ 4 \le i, j \le 5, \tag{10}$$

here $\tilde{F}^a_{ij}$s are components of the field strength $\tilde{\mathcal{F}}^a_{ij}$ in Eq. (7). Given the generalized field strength we first discuss the superspace formulation of Yang-Mills Lagrangian only and then will incorporate the gauge fixing and ghost terms into superspace formulation. Let's consider the superspace Lagrangian

$$\tilde{\mathcal{L}}(\tilde{\mathcal{A}}) = -\frac{1}{4}\tilde{g}^{ik}\tilde{g}^{jl}\tilde{\mathcal{F}}^a_{ij}\tilde{\mathcal{F}}^a_{kl}, \tag{11}$$

here the $\tilde{g}^{ij} = \tilde{g}^{-1}_{ij}$ hence $\tilde{g}^{ij} = g^{ij}$, $0 \le i, j \le 3$; $\tilde{g}^{45} = -\tilde{g}^{54} = 1 = \tilde{g}_{54}$. The Lagrangian is gauge invariant under a generalized gauge transformation in the Eq. (8). The Lagrangian in Eq. (11) can be expanded as

$$\begin{aligned}\tilde{\mathcal{L}}(\tilde{\mathcal{A}}) = &-\frac{1}{4}\tilde{F}^a_{\mu\nu}(X)\tilde{F}^{a\mu\nu}(X) - \tilde{F}^a_{4\mu}(X)\tilde{F}^a_{5\mu}(X) - \frac{1}{2}\tilde{F}^a_{44}(X)\tilde{F}^a_{55}(X)\\ &+ \frac{1}{2}\tilde{F}^a_{45}(X)\tilde{F}^a_{54}(X),\end{aligned} \tag{12}$$

here $0 \le \mu, \nu \le 3$ and the argument X signifies the superspace. The first term is the usual Yang-Mills Lagrangian except for the fact that it is defined on X. From the general definition in Eq. (7), we list out the components appearing in Eq. (12) as follows

$$\begin{aligned}\tilde{F}^a_{4\mu}(X) &= \partial_\theta\tilde{A}^a_\mu - \partial_\mu F^a_4 - gf^{abc}F^b_4\tilde{A}^c_\mu\\ &= \partial_\theta\tilde{A}^a_\mu - D_\mu F^a_4,\end{aligned} \tag{13}$$

the covariant derivative of F^a_4 as previously defined is $D_\mu F^a_4 = \partial_\mu F^a_4 - gf^{abc}A^b_\mu F^c_4$. Similarly, other components are

$$\tilde{F}^a_{5\mu}(X) = \partial_\lambda\tilde{A}^a_\mu - D_\mu F^a_5, \tag{14}$$

$$\tilde{F}^a_{55}(X) = 2\partial_\lambda F^a_5 - gf^{abc}F^b_5F^c_5, \tag{15}$$

$$\tilde{F}^a_{44}(X) = 2\partial_\theta F^a_4 - gf^{abc}F^b_4F^c_4, \tag{16}$$

$$\tilde{F}^a_{54}(X) = \partial_\lambda F^a_4 + \partial_\theta F^a_5 - gf^{abc}F^b_5F^c_4 = \tilde{F}^a_{45}(X). \tag{17}$$

Therefore, the Lagrangian in Eq. (12) is rewritten as follows

$$\tilde{\mathcal{L}}(\tilde{\mathcal{A}}) = -\frac{1}{4}\tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu} - (\partial_\theta \tilde{A}^a_\mu - D_\mu F^a_4)(\partial_\lambda \tilde{A}^a_\mu - D_\mu F^a_5) - \frac{1}{2}(2\partial_\theta F^a_4 - gf^{abc}F^b_4 F^c_4)(2\partial_\lambda F^a_5 - gf^{abc}F^b_5 F^c_5) + \frac{1}{2}(\partial_\lambda F^a_4 + \partial_\theta F^a_5 - gf^{abc}F^b_5 F^c_4)^2. \tag{18}$$

Now we make the following identifications

$$F^a_4 = F^a_5 = \tilde{c^a}(X) = c^a(x) + \theta\, c^a_\theta(x) + \lambda\, c^a_\lambda(x) + \theta\lambda\, c^a_{\theta\lambda}(x) \tag{19}$$

then in particular, we identify

$$\partial_\theta \tilde{A}^a_\mu = \partial_\lambda \tilde{A}^a_\mu = D_\mu \tilde{c^a}, \tag{20}$$

$$\partial_\lambda \tilde{c^a} = \partial_\theta \tilde{c^a} = \frac{1}{2} g f^{abc}\tilde{c^b}\tilde{c^c}. \tag{21}$$

Under the given identifications of Eqs. (20), (21) we see that superspace Lagrangian $\tilde{\mathcal{L}}(\tilde{\mathcal{A}})$ in Eq. (11) reduces to usual Yang-Mills Lagrangian in Eq. (2) apart from the fact that all fields are functions of X on account of the Jacobi identity

$$(f^{abc}\tilde{c^b}\tilde{c^c})^2 = 0. \tag{22}$$

We further observe that the two identifications of Eqs. (20), (21) are nothing but the BRST transformations of the gauge and the ghost field respectively. The $\tilde{c^a}$ thus relates to the ghost field. Let's expand the Eqs. (20), (21) into components to get (using Eqs. (5), (19))

$$A^a_{\mu,\theta}(x) = A^a_{\mu,\lambda}(x) = D_\mu c^a(x) \Longrightarrow A^a_{\mu,\theta\lambda} = 0, \tag{23}$$

$$c^a_\theta(x) = c^a_\lambda(x) = \frac{1}{2} f^{abc}c^b c^c \Longrightarrow c^a_{\theta\lambda} = 0. \tag{24}$$

Thus, as we argued, in the first approach, fields F^a_4 and F^a_5 are related to the ghost field and coefficients of θ and λ both of a given superfield are identified with BRST transformations. Thus, coordinates θ and λ act as a global anti-commuting parameter of the BRST transformation in the first approach. We are yet to introduce the anti-ghost field in the formulation which we do now to obtain the 6-dim superspace version of the gauge fixing and the ghost term. We consider the following superspace Lagrangian

$$\tilde{\mathcal{L}}_{gg} = \frac{\partial}{\partial\theta}\left[\tilde{\overline{c^a}}(X)\Big(\partial.\tilde{A}^a(X) + \frac{\xi}{2}\frac{\partial}{\partial\theta}\tilde{\overline{c^a}}(X)\Big)\right] \tag{25}$$

where $\tilde{\overline{c^a}}$ is the anti-ghost field in 6 dimensions expanded as

$$\tilde{\overline{c^a}} = -\overline{c^a}(x) + \theta\,\overline{c^a}_\theta(x) + \lambda\,\overline{c^a}_\lambda(x) + \theta\lambda\,\overline{c^a}_{\theta\lambda}(x). \tag{26}$$

We now inspect whether the Lagrangian in Eq. (2) leads to the gauge fixing and the ghost term of Eq. (2) under BRST transformations of Eqs. (23), (24). We expand the RHS of Eq. (25) as shown below

$$\tilde{\mathcal{L}}_{gg} = -\overline{c^a}(x)\partial.A^a_{,\theta}(x) + \overline{c^a}_\theta(x)\partial.A^a(x) + \frac{\xi}{2}(\overline{c^a}_\theta)^2 + \lambda\Big(\overline{c^a}_\lambda(x)\partial.A^a_{,\theta}(x) - \overline{c^a}(x)\partial.A^a_{,\theta\lambda}(x) + \overline{c^a}_{\theta\lambda}(x)\partial.A^a(x) + \overline{c^a}_\theta\partial.A^a_{,\lambda}(x)\Big), \tag{27}$$

where $\partial.A^a = \partial^\mu A^a_\mu$ and $\partial.A^a_{,\theta} = \partial^\mu A^a_{\mu,\theta}$, $(\mu = 0, ..., 3)$. The components $A^a_{,\theta}$, $A^a_{,\lambda}$ and $A^a_{,\theta\lambda}$ have already been computed in Eq. (23), putting them back in Eq. (27) we get

$$\tilde{\mathcal{L}}_{gg} = -\overline{c^a}(x)\partial^\mu D_\mu c^a + \overline{c^a}_\theta(x)\partial.A^a(x) + \frac{\xi}{2}(\overline{c^a}_\theta)^2 + \lambda\Big(\overline{c^a}_\lambda(x)\partial^\mu D_\mu c^a + \overline{c^a}_{\theta\lambda}(x)\partial.A^a(x) + \overline{c^a}_\theta\partial^\mu D_\mu c^a\Big). \tag{28}$$

Further we make the following identifications regarding the anti-ghost

$$\overline{c^a}_\theta(x) = \overline{c^a}_\lambda(x) = F^a(x) \text{ (Auxiliary field of Eq. (2))} \Longrightarrow \overline{c^a}_{\theta\lambda}(x) = 0. \tag{29}$$

This identification is precisely the BRST transformation of the anti-ghost field. In view of Eq. (29), we get

$$\tilde{\mathcal{L}}_{gg} = -\overline{c^a}\partial^\mu D_\mu c^a + \frac{\xi}{2}F^{a2} + F^a\partial.A^a(x) + 2\lambda\, F^a\partial^\mu D_\mu c^a.$$

The last term multiplying λ vanishes as a result of Eq. of motion of the anti-ghost $\overline{c^a}$ [1] and we see that superspace Lagrangian of Eq. (25) reduces to 4-dim gauge fixing and ghost term of Eq. (2) given identifications (BRST transformations) of gauge, ghost and anti-ghost fields as in Eqs. (20), (21), (29). The gauge fixing and ghost term in the superspace formulation has an elegant and simple form. Thus, the effective theory in Lorenz gauge (Eq. (2)) in the superspace is given by

$$\tilde{\mathcal{L}}_{superL} = -\frac{1}{4}\tilde{g}^{ik}\tilde{g}^{jl}\tilde{\mathcal{F}}^a_{ij}\tilde{\mathcal{F}}^a_{kl} + \frac{\partial}{\partial\theta}\left[\tilde{\overline{c^a}}(\partial.\tilde{A}^a + \frac{\xi}{2}\frac{\partial}{\partial\theta}\tilde{\overline{c^a}})\right], \quad (i, j, k, l = 0, ..., 5). \tag{30}$$

In the second approach discussed in the introduction, the gauge, ghost and anti-ghost form a supermultiplet in which gauge and anti-ghost fields of Eqs. (5), (26) are as follows [5–7]

$$\tilde{A}^a_\mu = A^a_\mu + \theta\, D_\mu c^a + \lambda\, D_\mu \overline{c^a} + \theta\lambda\,(D_\mu F^a - gf^{abc}D_\mu \overline{c^b}c^c), \tag{31}$$

$$\tilde{\overline{c^a}} = -\overline{c^a} + \theta\, F^a + \lambda\,\frac{g}{2}f^{abc}\overline{c^b}\overline{c^c} + \theta\lambda\, gf^{abc}F^b\overline{c^c}. \tag{32}$$

Therefore, the Lagrangian in the superspace is different and as follows [7]

$$\tilde{\mathcal{L}}_{superL} = -\frac{1}{4}\tilde{g}^{ik}\tilde{g}^{jl}\tilde{\mathcal{F}}^a_{ij}\tilde{\mathcal{F}}^a_{kl} + \frac{1}{2}\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\theta}\Big(\tilde{A}^a_\mu\tilde{A}^{\mu a}\Big) - \frac{\xi}{2}\Big(\frac{\partial}{\partial\theta}\tilde{\overline{c^a}}\Big)^2. \tag{33}$$

It is interesting to compare Eqs. (30) and (33). The former has a simpler form and relies only on BRST transformations although the later is more encompassing in its treatment. In the present theory, only BRST transformations are known and consistent though a bit unusual, therefore the first approach is chosen. Having discussed the setup, we next review the theory in the quadratic gauge with additional Lorenz gauge fixing for one of the gluons.

3. *SU*(*N*) QCD in the quadratic gauge

Here we discuss a model which we intend to formulate in the 6-dim superspace. The model relies on the new quadratic gauge fixing of Yang-Mills action. In its simplest form, the gauge is also free of Gribov ambiguity as it is purely an algebraic gauge. However this gauge is not suitable for usual perturbation theory. The new quadratic gauge has been introduced as follows [14],

$$H^a[A^\mu(x)] = A^a_\mu(x)A^{\mu a}(x) = f^a(x); \quad \text{for each } a \tag{34}$$

where $f^a(x)$ is an arbitrary function of x. The Faddeev-Popov determinant in this gauge is given by

$$\det\left(\frac{\delta(A_\mu^{a\epsilon}A^{\mu a\epsilon})}{\delta\epsilon^b}\right)=\det\left(2A_\mu^a(\partial^\mu\delta^{ab}-gf^{acb}A^{\mu c})\right). \quad (35)$$

Therefore, the resulting effective Lagrangian density contains gauge fixing and ghost terms as follows,

$$\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm ghost}=-\frac{1}{2\zeta}\sum_a(A_\mu^aA^{\mu a})^2-2\sum_a\overline{c^a}A^{\mu a}(D_\mu c)^a, \quad (36)$$

where ζ is an arbitrary gauge fixing parameter and $(D_\mu c)^a=\partial_\mu c^a-gf^{abc}A_\mu^b c^c$. Now onwards, we shall drop the summation symbol, but the summation over an index a will be understood when it appears repeatedly, including when repeated *thrice* as in the ghost terms above. In particular,

$$-\overline{c^a}A^{\mu a}(D_\mu c)^a=-\overline{c^a}A^{\mu a}\partial_\mu c^a+gf^{abc}\overline{c^a}c^cA^{\mu a}A_\mu^b \quad (37)$$

where the summation over indices a, b and c each runs independently over 1 to N^2-1. With this understanding, we write the full effective Lagrangian density in this quadratic gauge as

$$\begin{aligned}\mathcal{L}_Q&=-\frac{1}{4}F_{\mu\nu}^aF^{\mu\nu a}-\frac{1}{2\zeta}(A_\mu^aA^{\mu a})^2-2\overline{c^a}A^{\mu a}(D_\mu c)^a\\&=-\frac{1}{4}F_{\mu\nu}^aF^{\mu\nu a}+\frac{\zeta}{2}F^{a2}+F^aA_\mu^aA^{\mu a}-2\overline{c^a}A^{\mu a}(D_\mu c)^a. \quad (38)\end{aligned}$$

As shown in [15], the Lagrangian is BRST invariant [19,20] which is essential for the ghost independence of green functions and unitarity of the S-matrix.

4. Superspace version of QCD with a new and extended BRST

The non-zero ghost condensates instrumental for the main non perturbative result of the quadratic gauge was established with the help of additional gauge fixing for one of the gluons [14,17]. The presence of this additional Lorenz gauge fixing does reintroduce the Gribov ambiguity for this component but this is the compromise to be made for an explicit demonstration of effective masses for the off diagonal gluons. Our aim here is to formulate the superspace version of the effective theory in quadratic gauge with additional Lorenz gauge fixing for one of the gluons. It is a novel formulation as the theory involves two pairs ghosts and thus contains some peculiar BRST transformations. Earlier superspace formulations only included a pair ghosts $c,\bar{c}$ as we have seen in the previous section. We now write the effective action of the quadratic gauge with additional gauge fixing for one of the gluons which is as follows

$$\begin{aligned}\mathcal{S}_{eff}&=\mathcal{S}_Q+\int d^4x\left[\frac{\xi}{2}(G^3)^2+G^3\partial^\mu A_\mu^3-\overline{d^3}\partial^\mu(D_\mu d)^3\right] \quad (39)\\&=\mathcal{S}_Q+\int d^4x\left[\frac{\xi}{2}(G^3)^2+G^3\partial^\mu A_\mu^3-\overline{d^3}\Box d^3\right],\end{aligned}$$

where $\Box$ stands for $\partial^\mu\partial_\mu$ and a set of additional fields $G^3, d^3, \overline{d^3}$ correspond to the additional Lorenz gauge of the diagonal gluon A^3 and the ghosts $\overline{d^3}, d^3$ are treated as $SU(3)$ singlets.

As a first try it is easy to see that the action in Eq. (39) is invariant under the following nilpotent BRST transformations

$$\begin{aligned}\delta c^d&=\frac{\delta\omega}{2}f^{dbc}c^bc^c\\\delta\overline{c^d}&=\frac{\delta\omega}{g}F^d\\\delta A_\mu^d&=\frac{\delta\omega}{g}(D_\mu c)^d\\\delta F^d&=0\\\delta G^3&=0\\\delta\Box d^3&=0\\(\delta\overline{d^3})\Box d^3&=\frac{\delta\omega}{g}G^3\partial^\mu D_\mu c^3,\end{aligned} \quad (40)$$

where $\delta\omega$ infinitesimal, anticommuting and global parameter. This set of transformations differs from the usual in the second last in which a transformation of a differential operator acting on the field is defined. It can in general be solved for $\delta d^3(x)$ locally. The set also differs in the composite form of the last of the Eq. (40), which can always be defined locally in the same spirit of first five transformations. Now the question is that is it possible to have consistent superspace version which reduces to Eq. (39), given the transformations in Eq. (40)? The first four of transformations in Eq. (40) suggests that a formulation of the $\mathcal{S}_Q=\int d^4x\mathcal{L}_Q$ in Eq. (39) must be similar to the discussion in the previous section which we deal with later. The last three suggests that the fields which can appear in 6-dim formulation, if it exists, of the part other than $\mathcal{S}_Q$ in RHS of Eq. (39) are G^3, d^3 as the transformation of $\overline{d^3}$ is defined only compositely. Therefore, it is clear that we shall not get the superspace formulation which will reduce to the action in Eq. (39) in 4-dim, provided the transformations in Eq. (40) as transformations of G^3, d^3 are trivial. This gives a good lesson that not all available sets of BRST transformations lead to the consistent superspace version. This conclusion has never been evident from previous studies. Therefore, we need to search a new set of transformations in which δG^3 and δd^3 are non trivial. Let's consider the following transformations,

$$\begin{aligned}\delta c^d&=\frac{\delta\omega}{2}f^{dbc}c^bc^c\\\delta\overline{c^d}&=\frac{\delta\omega}{g}F^d\\\delta A_\mu^d&=\frac{\delta\omega}{g}(D_\mu c)^d\\\delta F^d&=0\\\delta G^3&=\frac{\delta\omega}{g}\overline{d^3}\\\delta\Box d^3&=\frac{\delta\omega}{g}(\partial_\mu A^{\mu 3}+\xi G^3)\\(\delta\overline{d^3})\Box d^3&=\frac{\delta\omega}{g}G^3\partial^\mu D_\mu c^3.\end{aligned} \quad (41)$$

As for the previous transformations, we observe that the second last of Eqs. (41) is inhomogeneous wave equation for BRST differential δd^3 with a simple form of the right hand side acting as the source, which surely admits a local solution for $\delta d^3(x)$ for given fields. Now this set of transformations has become applicable for superspace formulation as $\delta G^3, \delta\Box d^3$ are non zero. The action (39) is invariant under transformations of Eq. (41).

Thus, we are now in a position to formulate the theory in Eq. (39) in the superspace. First we define the superfields related to auxiliary field G^3 and a pair of ghosts $d^3, \overline{d^3}$ respectively as follows

$$\tilde{G}^3(X) = G^3(x) + \theta\, G^3_\theta(x) + \lambda\, G^3_\lambda(x) + \theta\lambda\, G^3_{\theta\lambda}(x) \tag{42}$$

$$\tilde{d}^3(X) = \Box d^3(x) + \theta\, d^3_\theta(x) + \lambda\, d^3_\lambda(x) + \theta\lambda\, d^3_{\theta\lambda}(x) \tag{43}$$

$$\tilde{\overline{d^3}}(X) = -\overline{d^3}(x) + \theta\, \overline{d^3}_\theta(x) + \lambda\, \overline{d^3}_\lambda(x) + \theta\lambda\, \overline{d^3}_{\theta\lambda}(x). \tag{44}$$

The $\Box$ operator in the first term of RHS in Eq. (43) should be noted. We shall use the same convention of the section 2 for fields $A, c, \bar{c}$. As discussed earlier, the superspace formulation of the theory given by $\mathcal{L}_Q$ alone must be similar to the formulation given in Sec. 2 as BRST transformations are the same in both cases and superspace version of the part other than $\mathcal{S}_Q$ in RHS of Eq. (39) should only include fields G^3, d^3. Keeping these considerations in sight, we examine the following simple ansatz for the superspace version of gauge fixing and ghost terms of both the quadratic gauge and Lorenz gauge for the gluon A^3 as in Eq. (39),

$$\tilde{\mathcal{L}}_{Q\,gg}(X) = \frac{\partial}{\partial\theta}\left[\tilde{\overline{c^a}}(X)\Big(\tilde{A}^{a\mu}(X)\tilde{A}^a_\mu(X) + \frac{\zeta}{2}\frac{\partial}{\partial\theta}\tilde{\overline{c^a}}(X)\Big) + \tilde{G^3}(X)\tilde{d^3}(X)\right]. \tag{45}$$

We bring to attention the new form of the second part which is different than a usual form, (anti-ghost $\times$ gauge condition). We proceed to check that given the transformations (41), whether the ansatz gives back usual 4-dim gauge fixing and ghost terms of Eq. (39) or not. Using Eqs. (5), (26), (42), (43), we expand Eq. (45) as follows

$$\begin{aligned}\tilde{\mathcal{L}}_{Q\,gg} = {}& -2\overline{c^a}A^{\mu a}A^a_{\mu,\theta} + \overline{c^a}_\theta A^{\mu a}A^a_\mu + \frac{\zeta}{2}(\overline{c^a}_\theta)^2 + G^3_\theta \Box d^3 + G^3 d^3_\theta \\ & + \lambda\Big(\overline{c^a}_\lambda A^{\mu a}A^a_{\mu,\theta} - \overline{c^a}A^{\mu a}A^a_{\mu,\theta\lambda} + \overline{c^a}_{\theta\lambda}A^{\mu a}A^a_\mu \\ & + \overline{c^a}_\theta A^{\mu a}A^a_{\mu,\lambda} + G^3 d^3_{\theta\lambda} + G^3_{\theta\lambda}\Box d^3 + G^3_\lambda d^3_\theta + G^3_\theta d^3_\lambda\Big).\end{aligned} \tag{46}$$

Since BRST transformations for fields $A, c, \bar{c}$ and F remains same, components $A^a_{\mu,\theta}, A^a_{\mu,\lambda}, A^a_{\mu,\theta\lambda}, \overline{c^a}_\theta, \overline{c^a}_\lambda, \overline{c^a}_{\theta\lambda}$ are readily known and given in Eqs. (23), (29). For the rest in line with what is done so far, considering transformations of G^3 and d^3 as in Eq. (41) we make following identifications

$$\partial_\theta \tilde{G}^3 = \partial_\lambda \tilde{G}^3 = \tilde{\overline{d^3}} \tag{47}$$

$$\partial_\theta \tilde{d}^3 = \partial_\lambda \tilde{d}^3 = (\partial_\mu \tilde{A^{\mu 3}} + \xi\tilde{G}^3) \tag{48}$$

The Eq. (47) implies the following

$$G^3_\theta = G^3_\lambda = -\overline{d^3}, \tag{49}$$

and Eq. (48) implies that

$$d^3_\theta = d^3_\lambda = (\partial_\mu A^{\mu 3} + \xi G^3),\; d^3_{\theta\lambda} = \partial^\mu D_\mu c^3 - \xi\overline{d^3},\; G^3_{\theta\lambda} = 0. \tag{50}$$

Everything is in order and as we want it to be except for the fact that $d^3_{\theta\lambda}$ is non zero. As long as it is non trivial we would get terms in addition to already existing terms in Eq. (39) which implies that the superspace theory does not reduce to Eq. (39). Therefore, the formulation is still not fully consistent given transformations (41). The progress so far suggests that we modify the transformation somehow rather than Eq. (45). The reason for $d^3_{\theta\lambda} \neq 0$ is that the transformation of $\Box d^3$ in Eq. (41) is not nilpotent but obey the following higher degree algebra [16]

$$\delta^2((\delta^3\Box d^3)\Box d^3)\Box d^3 = 0. \tag{51}$$

We introduce a novel trick to make the transformation nilpotent. We express the effective action (39) in terms of a new auxiliary field B^3,

$$\begin{aligned}\mathcal{S}_{eff} &= \mathcal{S}_Q + \int d^4x\left[\frac{\xi}{2}(G^3)^2 + G^3\partial^\mu A^3_\mu - \overline{d^3}\Box d^3\right] \\ &= \mathcal{S}_Q + \int d^4x\left[\frac{-1}{2\xi}(B^3)^2 + B^3G^3 + G^3\partial^\mu A^3_\mu - \overline{d^3}\Box d^3\right]\end{aligned} \tag{52}$$

The action (52) is invariant under following transformations

$$\begin{aligned}\delta c^d &= \frac{\delta\omega}{2} f^{dbc}c^b c^c \\ \delta\overline{c^d} &= \frac{\delta\omega}{g}F^d \\ \delta A^d_\mu &= \frac{\delta\omega}{g}(D_\mu c)^d \\ \delta F^d &= 0 \\ \delta G^3 &= \frac{\delta\omega}{g}\overline{d^3} \\ \delta B^3 &= -\frac{\delta\omega}{g}\partial^\mu D_\mu c^3 \\ \delta\Box d^3 &= \frac{\delta\omega}{g}(\partial_\mu A^{\mu 3} + B^3) \\ (\delta\overline{d^3})\Box d^3 &= \frac{\delta\omega}{g\xi}B^3\partial^\mu D_\mu c^3.\end{aligned} \tag{53}$$

We get the following nilpotent algebra [16]

$$\begin{aligned}&\delta^2 B^3 = 0 \\ &\delta^2\Box d^3 = 0.\end{aligned} \tag{54}$$

We see that the algebra of the transformation has been simplified significantly. Now we need to define a superfield for B^3 and because the transformation of B^3 is given we get

$$\tilde{B}^3(X) = B^3(x) + \theta\left(B^3_\theta(x) = -\partial^\mu D_\mu c^3\right) + \lambda\left(B^3_\lambda(x) = -\partial^\mu D_\mu c^3\right). \tag{55}$$

Owing to this new transformation, Eq. (48) modifies to

$$\begin{aligned}&\partial_\theta \tilde{d}^3 = \partial_\lambda \tilde{d}^3 = (\partial_\mu \tilde{A}^3_\mu + \tilde{B}^3) \Longrightarrow \\ &d^3_\theta = d^3_\lambda = (\partial_\mu A^{\mu 3} + B^3),\; d^3_{\theta\lambda} = 0,\; G^3_{\theta\lambda} = 0.\end{aligned} \tag{56}$$

The trick worked and we got $d^3_{\theta\lambda} = 0$. Putting components calculated in Eqs. (23), (29), (49), (56) back into Eq. (46), we get

$$\begin{aligned}\tilde{\mathcal{L}}_{Q\,gg} = {}& \frac{\zeta}{2}F^{a2} + F^a A^a_\mu A^{\mu a} - 2\overline{c^a}A^{\mu a}(D_\mu c)^a + G^3(\partial^\mu A^3_\mu + B^3) \\ & - \overline{d^3}\Box d^3 + 2\lambda\Big(F^a A^{\mu a}D_\mu c^a + \overline{d^3}(\partial_\mu A^{\mu 3} + B^3)\Big).\end{aligned} \tag{57}$$

The terms multiplying λ vanish in view of Eq. of motion of fields $\bar{c}$ and G^3 and we get

$$\begin{aligned}\tilde{\mathcal{L}}_{Q\,gg} = {}& \frac{\zeta}{2}F^{a2} + F^a A^a_\mu A^{\mu a} - 2\overline{c^a}A^{\mu a}(D_\mu c)^a \\ & + G^3(\partial^\mu A^3_\mu + B^3) - \overline{d^3}\Box d^3.\end{aligned} \tag{58}$$

Thus, the superspace Lagrangian has reduced to usual 4-dim Lagrangian of Eq. (52) consistently except for the redundant term representing the background energy of the auxiliary field B^3 when transformations (53) are applied. Hence, we write finally the superspace Lagrangian for the full theory in Eq. (52) as follows

$$\tilde{\mathcal{L}}_{superQ}(X) = -\frac{1}{4}\tilde{g}^{ik}\tilde{g}^{jl}\tilde{\mathcal{F}}^{a}_{ij}\tilde{\mathcal{F}}^{a}_{kl} + \frac{\partial}{\partial\theta}\left[\tilde{\bar{c}^{a}}\left(\tilde{A}^{a\mu}\tilde{A}^{a}_{\mu} + \frac{\zeta}{2}\frac{\partial}{\partial\theta}\tilde{\bar{c}^{a}}\right) + \tilde{G^{3}}\tilde{d^{3}}\right], \quad (59)$$

which is fully consistent only with transformations in Eq. (53). The importance of nilpotency of the BRST transformation in the superspace formulation has been made explicit. The result is encouraging as it will have implications on the Ward identity in the superspace for the following reason. The present superspace theory is very simple being linear in superfields $\tilde{G^3}, \tilde{d^3}$. It also contains less number of fields than in 4-dim theory of Eq. (52). Therefore, it would be interesting to know the structure of the Ward identity in the superspace which is expected to be simpler than that in 4 dimensions.

5. Conclusion

In some of the recent earlier work the interesting features of a purely quadratic gauge condition have been studied and shown to lead to several interesting characteristics in the non perturbative sector of QCD. Superspace formulation is an elegant method in which a gauge theory takes a simple form in the extended space with additional Grassmannian dimensions. In this paper as given in Eq. (59), we developed a new formulation of the effective theory having two pairs of ghosts and some unconventional transformations. We encountered several technical ambiguities in the process due to a unique structure of the theory. The solutions of which provided novel results. In particular, we saw that superspace formulation may not be consistent under all available sets of BRST transformations of the same theory i.e., the theory in superspace does not reduce to corresponding 4-dim theory under all sets of transformations. We have come across an exceptional application of an auxiliary field in restoring the nilpotency of the transformations. We further made explicit the importance of nilpotency of transformations in the superspace formulation. The simple structure of Eq. (59) hints at a simple form of superspace Ward identities of the theory which we intend to explore soon.

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