Research Article
String Theory Explanation of Galactic Rotation Found Using the Geodesic Constraint

Mark D. Roberts 🔘

Burpham Institute of Advanced Study, UK

Correspondence should be addressed to Mark D. Roberts; robemark@gmail.com

The unique spherically symmetric metric which has vanishing Weyl tensor, is asymptotically de-Sitter, and can model constant galactic rotation curves is presented. Two types of field equations are shown to have this metric as an exact solution. The first is Palatini varied scalar-tensor theory. The second is the low energy limit of string theory modified by inclusion of a contrived potential.

1. Introduction

Galactic rotation curves often exhibit speeds which are a constant independent of distance from the center of the galaxy. This is less than what would be expected from solid body rotation where the rotation speed increases with radial distance and more than what would be expected from free orbit rotation where the speed would decrease with radial distance. Currently the majority view is that a large amount of "dark matter" occurs in nonluminous places to produce these rotation curves, as opposed to the minority view which is that constant rotation curves are caused by modified laws of gravity. Examples of modified gravity used to explain the rotation curves are Born-Infeld gravity [1] and noncommutative geometry [2]. From a Newtonian perspective the gravitational modification which works is the replacement of the Newtonian reciprocal gravitational potential by a logarithmic potential; the spherically symmetric relativistic generalization of this [3] has one free function which in the present work is fixed by requiring that the Weyl tensor vanishes. This leaves the problem of finding which field equations the Einstein tensor obeys and both Palatini varied scalar-tensor theory [4] and the low energy limit of string theory with added potential are found to work. The usual method of approaching problems is by starting with a Lagrangian, then deriving field equations, finding solutions, and finally comparing with observations; here this

is reversed: in the present case it is observation, then metric, then field equations, and finally Lagrangian. This approach was first suggested in [5] where it was first mentioned that constant rotation curves constrain the geodesics; the actually equations were finally found in [6].

2. The Metric

Consider the line element

$$ds^2 = -\left(X + \lambda_t r^2\right) dt^2 + \frac{\left(X - v^2\right)^2}{X\left(X + \lambda_r r^2\right)} dr^2 + r^2 d\Sigma_2^2, \quad (1)$$

where

$$X \equiv \left(1 + 2v^2 \ln(r)\right),$$
$$d\Sigma_2^2 \equiv d\theta^2 + \sin(\theta)^2 d\phi^2, \quad (2)$$

and v is the constant speed of galactic rotation. $g_{\theta\theta}$ and g_{tt} are fixed up to a radial coordinate transformation by requiring constant rotation curves and g_{rr} is fixed by requiring that the Weyl tensor vanishes. When $\lambda_t(\lambda_t - \lambda_r) = 0$ the Weyl tensor vanishes; from now on only consider the solution $\lambda = \lambda_t = \lambda_r$. Another property of the curvature is that the Ricci scalar obeys a type of conformal wave equation $R = 6r\Box(1/r)$, but there appears to be no pattern to the higher

order Ricci curvature invariants. The curvature of the metric (1) is characterized by the Ricci scalar and

$$
Y \equiv R^t{}_t - R^\theta{}_\theta = \frac{2v^2}{r^2 (X - v^2)},
$$

$$
Z \equiv 3R^t{}_t - R^\theta{}_\theta = \frac{4v^6}{r^2 (X - v^2)^3}.
$$

(3)

The metric (1) has vanishing Weyl tensor so that it can be expressed in conformally flat form $d\hat{s}^2 = \Omega^2 ds^2_{minkowski}$, taking $\lambda = 0$ and comparing the $g_{\theta\theta}$ and g_{rr} terms, respectively,

$$
\Omega^2 R^2 = r^2,
$$

$$
R = \frac{Ar}{\sqrt{X}},
$$

(4)

where A is a constant of integration; to invert the second of (4) requires lambert W functions. Together with (7), (4) implies

$$
\Omega = \pm\frac{r}{R} = \pm\frac{1}{A}\sqrt{X} = \pm\frac{1}{A}\exp\left(-\frac{\alpha\phi}{2}\right).
$$

(5)

Thus the field equations (8) can be expressed in terms of the conformal factor Ω and can be thought of as the difference in the Ricci tensor between $d\hat{s}$ and ds. This implies that any metric of the form $d\hat{s}^2 = \Omega^2 ds^2_{ricciflat}$ will also be a solution of (8). In particular transforming the Schwarzschild solution

$$
ds^2 = -X\left(1 - \frac{2m\sqrt{X}}{r}\right)dt^2 + \frac{(X - v^2)^2}{X^2\left(1 - 2m\sqrt{X}/r\right)}dr^2
$$

$$
+ r^2 d\Sigma_2^2
$$

(6)

and this is still a solution of (8) with scalar field given by (7).

3. Palatini Scalar-Tensor Theory

Choosing the scalar field

$$
\phi = -\frac{1}{\alpha}\ln(X),
$$

(7)

a tensor which vanishes is

$$
Q_{ab} \equiv R_{ab} - \alpha\phi_{;ab} + \frac{\alpha^2}{2}\phi_a\phi_b
$$

$$
- \frac{1}{2}g_{ab}\left\{\alpha\Box\phi + \alpha^2 (\nabla\phi)^2 - 6\lambda\exp(\alpha\phi)\right\},
$$

(8)

where α is an absorbable constant. The field equations expressed in terms of the Ricci tensor of Palatini varied scalar-tensor theory are

$$
8\pi\kappa^2 S_{ab} = \mathscr{A} R_{ab} - \mathscr{A}'\phi_{;ab} - \mathscr{B}_\mathscr{A}\phi_a\phi_b
$$

$$
- \frac{1}{2}g_{ab}\left\{\mathscr{A}'\Box\phi + \mathscr{A}'' (\nabla\phi)^2 + V(\phi)\right\},
$$

(9)

where

$$
\mathscr{B}_\mathscr{A} = \mathscr{B} + \mathscr{A}'' - \frac{3}{2}\frac{\mathscr{A}'^2}{\mathscr{A}},
$$

(10)

$\mathscr{A}$ is called the primary dilation function, $\mathscr{B}$ is called the secondary dilation function, and V is called the potential. The field equations (8) are a particular case of the field equations (9) with

$$
\mathscr{A} = \exp(\alpha\phi),
$$

$$
V = -6\lambda\exp(2\alpha\phi),
$$

$$
\mathscr{B} = 0,
$$

$$
S_{ab} = 0.
$$

(11)

The field equations (9) can be found by performing both metric and Palatini variations of the action

$$
S = \int d^4x \sqrt{-g}\left\{\mathscr{A}(\phi) R - \mathscr{B}(\phi)(\nabla\phi)^2 - V(\phi)\right\}.
$$

(12)

4. Low Energy String Theory with a Potential

The Lagrangian for low energy string theory with a potential is

$$
\mathscr{L} = \exp(-2\phi)\left\{R + \frac{\alpha}{2}\Box\phi - \beta(\nabla\phi)^2 - \frac{\gamma}{12}H^2 - V(\phi)\right\},
$$

$$
H^2 \equiv H_{abc}H^{abc},
$$

(13)

performing metric variation and then expressing the field equations in terms of the Ricci tensor

$$
8\pi\kappa^2 \exp(2\phi) S_{ab} = R_{ab} - \alpha\phi_{;ab} + 2\beta\phi_a\phi_b + \frac{\gamma}{2}H_{acd}H_b{}^{..cd}
$$

$$
+ g_{ab}\left(-\frac{\gamma}{6}H^2 + V\right),
$$

(14)

the metric (1) is a solution with scalar field given by (7) and

$$
H^{abc} = \pm\sqrt{\frac{\alpha^2 - 4\beta}{2\gamma}}\epsilon^{abcd}\phi_d,
$$

$$
V = \frac{v^2}{r^2 (X - v^2)^3}\left(X(X - 4v^2) + v^4\right).
$$

(15)

5. Properties and Comments

Seventeen properties and comments follow.

Firstly Seven Points on the Metric. M1: (1) has constant rotation curves when λr is negligible; the easiest way to see this is that the circular vector

$$
P^a = \left(\frac{v}{r}\delta^a{}_\theta + \delta^a{}_t\right)f(r),
$$

(16)

where f is an arbitrary function of r and has acceleration

$$
\dot{P}_a = \lambda r f^2 \delta^r_a,
$$

(17)

so that when λ vanishes the vector is acceleration-free or geodesic. For (6) with the same vector (16)

$$\dot{P}_a = m\sqrt{X}\left(X - 3v^2\right)\frac{f^2}{r^2}\delta_a^r, \tag{18}$$

so that when m vanishes the vector (16) is again geodesic.

M2: (1) is asymptotically de-Sitter as λr^2 increases much faster than $\ln(r)$. For $\lambda = 0$ the line element is not asymptotically flat as the ln term in g_{tt} diverges; this is a problem with many models of galactic rotation which have no natural long radial distance cut-off; having an asymptotically de-Sitter spacetime provides such a cut-off. Whether this can be thought of as evidence of a nonvanishing cosmological constant or just an indication of the effect of distant matter does not have to be chosen.

M3: the short distance cut-off for the metric is good; for short distances (6) shows that the galactic metric can approach Schwarzschild spacetime.

M4: at $r = 1$, $\ln(r)$ changes sign and it is not immediate where this is in meters; however note that the solution is still a solution with $r \longrightarrow r/r_o$ so that the length scale r_o is arbitrary and has to be fixed by other means.

M5: it is not clear what, if anything, corresponds to the vanishing of the three metric functions in (1); $(X - v^2)$ occurs in the denominator of curvature invariants so as it approaches zero they diverge.

M6: the line elements (1) and (6) were taken to be spherically symmetric rather than axisymmetric, but rotation of galactic spacetime would be expected and spacetime rotation would need a more elaborate model; one could simply choose $d\tilde{s}^2 = ds^2_{kerr}$, however Kerr rotation is fundamentally short range whereas galactic rotation is long range; the simpler case of the Newtonian model uses just the log potential $v^2\ln(r)$ so that the Newtonian model is spherically symmetric and this suggests that the simplest relativistic models are also spherically symmetric.

M7: why choose a line element with vanishing Weyl tensor in the first place: from the perspective of the Jordan formulation of Einstein's equations one might expect that at large distances the Weyl scalar is larger than the Ricci scalar; however from the perspective of the Schwarzschild solution which has vanishing Ricci tensor and Robertson-Walker spacetime which has vanishing Weyl tensor one might expect that vanishing Weyl tensor characterizes large distances; then the question arises as to at what range the Weyl tensor becomes nonnegligible; presumably this depends on the distance r_o introduced above, so far there the metric (6) suggests that it is a universal length rather than a length dependent on the mass of the galaxy under consideration.

Secondly Four Points on the Palatini-Scalar-Tensor Solution. *P1*: Palatini variations work well for general relativity where they reproduce the Christoffel connection, but not so for quadratic action theories where they act back on the Lagrangian to produce tensors which appear to be unconnected to anything: for scalar-tensor theories they act back on the primary dilation function producing a non-Christoffel connection and this turns out to be necessary in the present case; purely metric variation of the action (12) is unlikely to recover the field equations (8) as can be seen by subtracting off the θ, θ component from the t, t component: for the Palatini case the nonvanishing of Y in (3) can be matched to the $\phi_{;ab}$ term; however, for the purely metric case there is no $\phi_{;ab}$ term.

P2: the secondary dilation function $\mathscr{B}$ is taken to vanish (11); this means that there is no explicit kinetic term in (12), although there is an implicit kinetic term after Palatini variation; no explicit kinetic term is similar to some inflationary models where the potential V is related to the Hubble constant and the kinetic term is less important.

P3: as $k = 0$ Robertson-Walker cosmology is conformally flat the field equations (8) are obeyed for any choice of the scale factor; however, the field equations (9) and (11) might have as yet unexamined solutions and properties.

P4: because the scalar-tensor theory involves Palatini variations the underlying geometry is no longer Riemannian but rather Weyl with object of nonmetricity related to the primary dilation function $\mathscr{A}$, see [4].

Thirdly Three Points on the Properties of the String Solution (15). *S1*: the values of the constants in (15) lead to a real three-form H; they could have led to a complex one; although there are three constants after absorption there is only one free parameter.

S2: the potential in (15) is contrived; perhaps with the inclusion of additional fields it will no longer be necessary.

S3: the Lagrangian (13) includes a $\Box\phi$ term, as for Palatini scalar-tensor theory this term is necessary in order for the field equation involving Y in (3) to vanish; however the term can be removed from the Lagrangian by integration by parts.

Fourthly Four General Comments. *G1*: the introduction of the absorbable constant α is done because in the better known case of spherically symmetric static minimal scalar fields the value of the constant in the equation analogous to (7) depends on Schwarzschild mass, but the value of the constant in (8) does not, so it is anticipated that something similar could happen here.

G2: the equations (8) are assumed to be consistent as they can be Lagrangian based; however, conservation equations and the initial value problem are not looked at: conservation and Euler equations are not immediate in Palatini scalar-tensor theory; also the initial value problems are not straightforward when there are $\phi_{;ab}$ terms involved; these consistency problems depend on the variables and connection in which they are formulated.

G3: it is not immediate how the result impacts other areas of physics; for example, globular star clusters also exhibit unusual dynamics but usually have no overall v as in the present case; so far the post-Newtonian approximation of (1) has not been studied.

G4: potentials for cosmology are discussed in Calcagni [7]§7.3

6. Conclusion

A relativistic model of galactic rotation curves was produced which has the properties discussed in the last section. A problem with the metric is how many field equations and Lagrangians can it be a solution to. The most important problem with the scalar-tensor theory is the physical origin of the scalar field and the most likely explanation is that the scalar field comes from dimensional reduction. The most important problem with the string theory solution is the contrived potential which hopefully will at some time in the future be replaced by fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

References

[1] T. Harko, F. S. N. Lobo, M. K. Mak, and S. V. Suchkar, "Dark matter density profile and galactic metric in Eddington-inspired Born-Infeld gravity," *Modern Physics Letters A*, vol. 12, no. 9, Article ID 1450049, 2014.

[2] F. Rahaman, P. K. Kuhfittig, K. Chakraborty, A. A. Usmani, and S. Ray, "Galactic rotation curves inspired by a noncommutative-geometry background," *General Relativity and Gravitation*, vol. 44, no. 4, pp. 905–916, 2012.

[3] M. D. Roberts, "Halo spacetime," *Electronic Journal of Theoretical Physics*, vol. 8, pp. 299–310, 2011.

[4] M. D. Roberts, "Is spacetime non-metric?" *Expert Opinion on Astronomy and Astrophysics*, vol. 2, no. 1, 2018.

[5] M. D. Roberts, "A new approach to a relativistic model of galaxies," *Astrophysics and Space Science*, vol. 119, pp. 405–407, 1986.

[6] M. D. Roberts, "Galactic metrics," *General Relativity and Gravitation*, vol. 36, no. 11, pp. 2423–2431, 2004.

[7] G. Calcagni, *Classical and Quantum Cosmology*, Springer, Berlin, Germany, 2017.