

## Determination of the quantum numbers of $\Sigma_b(6097)^\pm$ via their strong decays

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(Received 14 November 2018; published 8 May 2019)

Recent experimental progress has led to the detection of many new hadrons. Very recently, the LHCb Collaboration announced the observation of two new  $\Sigma_b(6097)^\pm$  states in the  $\Lambda_b^0\pi^\pm$  invariant mass distribution, which are considered to be the excited states of the ground-state  $\Sigma_b^{(*)}$  baryon. Although almost all of the ground-state baryons have been observed, the fact that only a limited number of excited states have been observed makes them intriguing. Understanding the properties of the excited baryons would improve our knowledge about the strong interaction, as well as the nature and internal structures of these baryons. To specify the quantum numbers of the  $\Sigma_b(6097)^\pm$  states, we analyze their strong decays to  $\Lambda_b^0$  and  $\pi^\pm$  within the light-cone QCD sum rules formalism. To this end, they are considered as possible  $1P$  or  $2S$  excitations of either the ground-state  $\Sigma_b$  baryon with  $J = \frac{1}{2}$  or the  $\Sigma_b^*$  baryon with  $J = \frac{3}{2}$ , and their corresponding masses are calculated. The results of the analyses indicate that the  $\Sigma_b(6097)^\pm$  baryons are excited  $1P$  baryons with quantum numbers  $J^P = \frac{3}{2}^-$ .

DOI: [10.1103/PhysRevD.99.094003](https://doi.org/10.1103/PhysRevD.99.094003)

### I. INTRODUCTION

In the quark model, the heavy baryons containing one heavy and two light quarks form multiplets using the symmetries of flavor, spin, and spatial wave functions [1]. These considerations lead to the result that they belong to the sextet and antitriplet representations of  $SU(3)$ . At present, almost all of the ground-state heavy baryons have been observed in experiments. According to the quark model predictions, in addition to the ground states, the existence of their excited states is also expected. So far, only a few excited baryons have been observed in the bottom sector [2–6]. Detailed studies of the experimentally discovered states and searches for new, yet-to-be-observed states can play a critical role in our understanding of the internal structures of these states and give essential information about the dynamics of QCD in the nonperturbative domain.

Very recently, the LHCb Collaboration announced the first observation of two resonances  $\Sigma_b(6097)^-$  and  $\Sigma_b(6097)^+$  with masses  $m(\Sigma_b(6097)^-) = 6098.0 \pm 1.7 \pm 0.5$  and  $m(\Sigma_b(6097)^+) = 6095.8 \pm 1.7 \pm 0.4$  MeV [7]. The widths of these states were also measured as  $\Gamma(\Sigma_b(6097)^-) = 28.9 \pm$

$4.2 \pm 0.9$  and  $\Gamma(\Sigma_b(6097)^+) = 31.0 \pm 5.5 \pm 0.7$  MeV. Following the discovery of these states, the determination of their quantum numbers remains a central problem. To understand the structure of  $\Sigma_b(6097)$ , the authors of Ref. [8] performed mass and strong decay analyses within a quasi-two-body treatment. As a result of this study,  $\Sigma_b(6097)$  was concluded to be a bottom baryon candidate with  $J^P = \frac{3}{2}^-$  or  $J^P = \frac{5}{2}^-$ . In another study, the constituent quark model was applied to investigate  $\Sigma_b(6097)$ . The authors concluded that this state is a  $P$ -wave baryon with the quantum numbers  $J^P = \frac{3}{2}^-$  or  $J^P = \frac{5}{2}^-$  [9]. Another prediction for the quantum numbers of the observed  $\Sigma_b(6097)^-$  and  $\Sigma_b(6097)^+$  states was presented in Ref. [10] via the quark-pair creation model, which again indicated the possibility of either  $J^P = \frac{3}{2}^-$  or  $J^P = \frac{5}{2}^-$ .

In the present study, the properties of these baryons are studied in the framework of the QCD sum-rule method [11]. In our calculations, the observed states are considered as  $1P$  or  $2S$  excited states with  $J = \frac{1}{2}$  or  $J = \frac{3}{2}$ . We analyze the  $\Sigma_b^\pm \rightarrow \Lambda_b\pi^\pm$  decays and compare the values of the obtained decay widths with the experimental results, which allows us to determine the quantum numbers of the  $\Sigma_b(6097)^\pm$  states. To calculate the decay widths the main ingredients are the coupling constants corresponding to the considered transitions. To calculate these coupling constants we use the light-cone QCD sum rules (LCSR) method [12]. In this work, we also calculate the masses and decay constants of the states under consideration by

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taking into account all possibilities, i.e., assuming that these states are  $1P$  or  $2S$  excited states of the ground state  $\Sigma_b$  and  $\Sigma_b^*$  baryons with  $J = \frac{1}{2}$  or  $J = \frac{3}{2}$ . The obtained masses and decay constants are used as inputs in the numerical computations of the strong coupling constants of the related decays. Similar coupling constants for the ground-state baryons with a single heavy quark with  $J = \frac{1}{2}$  and  $J = \frac{3}{2}$  have been calculated in Refs. [13–16].

The paper is organized as follows. In Sec. II, the strong decays  $\Sigma_b(6097)^\pm \rightarrow \Lambda_b^0 \pi^\pm$  are studied within the LCSR method [12] by taking into account the possible configurations assigned to the  $\Sigma_b(6097)^\pm$  states. In this section, we also formulate the sum rules for the masses and decay constants of  $\Sigma_b(6097)^\pm$  with  $J = \frac{1}{2}$  or  $J = \frac{3}{2}$ . The numerical results of the masses and decay constants are used as input parameters in the analyses of the strong coupling constants defining the above strong decay channels. The numerical results of the strong coupling constants are also used to obtain the numerical values of the decay widths of the transitions under consideration. The last section contains our concluding remarks. The details of the calculations of the spectral densities are given in the Appendix.

## II. ANALYSIS OF THE $\Sigma_b \Lambda_b \pi$ VERTEX VIA LIGHT-CONE QCD SUM RULES

In this section, we analyze the strong transitions of the  $\Sigma_b(6097)^\pm$  states to the  $\Lambda_b^0$  and  $\pi^\pm$  particles. As we have already noted, our primary goal is to determine the quantum numbers of the recently observed  $\Sigma_b(6097)^\pm$  baryons. To this end, we assume that these states are  $1P$  or  $2S$  excitations of the corresponding ground-state baryons with  $J = \frac{1}{2}$  or  $J = \frac{3}{2}$ . We calculate the widths of these baryons under these assumptions and compare our results with the experimental data.

Each decay is characterized by its own strong coupling constant. Therefore, in the first step, we calculate the corresponding coupling constant defining the strong  $\Sigma_b \rightarrow \Lambda_b \pi$  transition for each case within the LCSR. For the ground-state  $\Sigma_b$  and  $\Sigma_b^*$  particles, these strong coupling constants are defined as

$$\begin{aligned} \langle \pi(q) \Lambda_b(p, s) | \Sigma_b(p', s') \rangle &= g_{\Sigma_b \Lambda_b \pi} \bar{u}(p, s) \gamma_5 u(p', s'), \\ \langle \pi(q) \Lambda_b(p, s) | \Sigma_b^*(p', s') \rangle &= g_{\Sigma_b^* \Lambda_b \pi} \bar{u}(p, s) u_\mu(p', s') q^\mu. \end{aligned} \quad (1)$$

For their corresponding  $1P$  and  $2S$  excitations, similar definitions as in Eq. (1) are used with the following replacements.

$$\begin{aligned} a) \text{ For the } 1P \text{ excitations: } & g_{\Sigma_b \Lambda_b \pi} \rightarrow g_{\Sigma_{b1} \Lambda_b \pi}, \quad g_{\Sigma_b^* \Lambda_b \pi} \rightarrow g_{\Sigma_{b1}^* \Lambda_b \pi}, \\ & u(p', s') \rightarrow \gamma_5 u(p', s'), \quad u_\mu(p', s') \rightarrow \gamma_5 u_\mu(p', s'), \\ & |\Sigma_b(p', s') \rangle \rightarrow |\Sigma_{b1}(p', s') \rangle, \quad \text{and} \quad |\Sigma_b^*(p', s') \rangle \rightarrow |\Sigma_{b1}^*(p', s') \rangle, \\ b) \text{ For the } 2S \text{ excitations: } & g_{\Sigma_b \Lambda_b \pi} \rightarrow g_{\Sigma_{b2} \Lambda_b \pi}, \quad g_{\Sigma_b^* \Lambda_b \pi} \rightarrow g_{\Sigma_{b2}^* \Lambda_b \pi}, \\ & |\Sigma_b(p', s') \rangle \rightarrow |\Sigma_{b2}(p', s') \rangle, \quad \text{and} \quad |\Sigma_b^*(p', s') \rangle \rightarrow |\Sigma_{b2}^*(p', s') \rangle. \end{aligned}$$

In this section and in all of the following discussions, the ground state and its  $1P$  and  $2S$  excitations are denoted by  $\Sigma_b$  ( $\Sigma_b^*$ ),  $\Sigma_{b1}$  ( $\Sigma_{b1}^*$ ), and  $\Sigma_{b2}$  ( $\Sigma_{b2}^*$ ) for the corresponding  $J = \frac{1}{2}$  ( $\frac{3}{2}$ ) baryons, respectively. Here,  $u(q, s)$  and  $u_\mu(q, s)$  are spinors corresponding to the  $J = \frac{1}{2}$  and  $J = \frac{3}{2}$  states, respectively.

To determine the aforementioned coupling constants from the LCSR, we introduce the following vacuum into the pseudoscalar meson correlation function:

$$\Pi_{(\mu)}(q) = i \int d^4 x e^{iq \cdot x} \langle \pi(q) | T \{ \eta_{\Lambda_b}(x) \bar{\eta}_{\Sigma_b^*(\mu)}(0) \} | 0 \rangle, \quad (2)$$

where the on-shell  $\pi$ -meson state is represented by  $\langle \pi(q) |$  with momentum  $q$ ,  $\eta_{\Sigma_b^*(\mu)}$  is used to represent the interpolating current of  $\Sigma_b^\pm$  ( $\Sigma_b^{\pm*}$ ), and  $\eta_{\Lambda_b}$  is the interpolating current for the  $\Lambda_b$  baryon with  $J = \frac{1}{2}$ . The interpolating fields for the  $J = \frac{1}{2}$  particles are given as

$$\begin{aligned} \eta_{\Lambda_b} &= \frac{1}{\sqrt{6}} \epsilon^{abc} \{ 2(u_a^T C d_b) \gamma^5 b_c + 2\beta(u_a^T C \gamma^5 d_b) b_c \\ &\quad + (u_a^T C b_b) \gamma^5 d_c + \beta(u_a^T C \gamma^5 b_b) d_c + (b_a^T C d_b) \gamma^5 u_c \\ &\quad + \beta(b_a^T C \gamma^5 d_b) u_c \}, \end{aligned} \quad (3)$$

and

$$\begin{aligned} \eta_{\Sigma_b} &= \frac{1}{\sqrt{2}} \epsilon^{abc} \{ (q_a^T C b_b) \gamma^5 q_c + \beta(q_a^T C \gamma^5 b_b) q_c \\ &\quad - (b_a^T C q_b) \gamma^5 q_c - \beta(b_a^T C \gamma^5 q_b) q_c \}. \end{aligned} \quad (4)$$

For the states with  $J = \frac{3}{2}$ , we have

$$\begin{aligned} \eta_{\Sigma_{b\mu}} &= \sqrt{\frac{1}{3}} \epsilon^{abc} \{ (q_a^T C \gamma_\mu q_b) b_c + (q_a^T C \gamma_\mu b_b) q_c \\ &\quad + (b_a^T C \gamma_\mu q_b) q_c \}. \end{aligned} \quad (5)$$

In the above equations,  $q$  is the  $u(d)$  quark field for  $\Sigma_b^{(*)+}$  ( $\Sigma_b^{(*)-}$ ). The indices  $a, b$ , and  $c$  represent the colors,  $C$  is the charge-conjugation operator, and  $\beta$  is an arbitrary mixing parameter. This mixing parameter is introduced to include all of the possible quark configurations in the interpolating currents considering the quantum numbers of the particles under consideration in order to write the possible general forms of the interpolating currents for the particles with  $J = \frac{1}{2}$ . The case  $\beta = -1$  corresponds to the Ioffe current.

To obtain the sum rules for the strong coupling constants we start with the standard procedure of QCD sum rules derivations. To obtain the physical or phenomenological sides of the desired sum rules, we insert complete sets of the  $\Sigma_b$  ( $\Sigma_b^*$ ) and  $\Lambda_b$  baryons into the correlation function. As a result, we get

$$\begin{aligned} \Pi_{(\mu)}^{\text{Phys}}(p, q) &= \frac{\langle 0 | \eta_{\Lambda_b} | \Lambda_b(p, s) \rangle \langle \pi(q) \Lambda_b(p, s) | \Sigma_b^{(*)}(p', s') \rangle \frac{\langle \Sigma_b^{(*)}(p', s') | \bar{\eta}_{\Sigma_b^{(*)}(\mu)} | 0 \rangle}{p'^2 - m^{(*)2}}}{p^2 - m_{\Lambda_b}^2} \\ &+ \frac{\langle 0 | \eta_{\Lambda_b} | \Lambda_b(p, s) \rangle \langle \pi(q) \Lambda_b(p, s) | \Sigma_{b1}^{(*)}(p', s') \rangle \frac{\langle \Sigma_{b1}^{(*)}(p', s') | \bar{\eta}_{\Sigma_{b1}^{(*)}(\mu)} | 0 \rangle}{p'^2 - m_1^{(*)2}} + \dots, \end{aligned} \quad (6)$$

$$\begin{aligned} \Pi_{(\mu)}^{\text{Phys}}(p, q) &= \frac{\langle 0 | \eta_{\Lambda_b} | \Lambda_b(p, s) \rangle \langle \pi(q) \Lambda_b(p, s) | \Sigma_b^{(*)}(p', s') \rangle \frac{\langle \Sigma_b^{(*)}(p', s') | \bar{\eta}_{\Sigma_b^{(*)}(\mu)} | 0 \rangle}{p'^2 - m^{(*)2}}}{p^2 - m_{\Lambda_b}^2} \\ &+ \frac{\langle 0 | \eta_{\Lambda_b} | \Lambda_b(p, s) \rangle \langle \pi(q) \Lambda_b(p, s) | \Sigma_{b2}^{(*)}(p', s') \rangle \frac{\langle \Sigma_{b2}^{(*)}(p', s') | \bar{\eta}_{\Sigma_{b2}^{(*)}(\mu)} | 0 \rangle}{p'^2 - m_2^{(*)2}} + \dots, \end{aligned} \quad (7)$$

where  $p$  is the momentum of the baryon  $\Lambda_b$  and  $p' = p + q$  is the momentum of the considered  $\Sigma_b^{(*)}$  and  $\Sigma_{bi}^{(*)}$  initial states, with  $i = 1$  or  $2$  indicating the  $1P$  or  $2S$  excited state. The dots at the ends of the equations are used to represent the contributions of the higher states and the continuum. It is well known that the physical (hadronic) side of the correlation function is complicated by the appearance of contributions from the baryonic states of both positive and negative parities. Constructing QCD sum rules for physical quantities free of the interference of unwanted (opposite) parity partners is of great importance (see Ref. [17] for more details). In our case, the hadronic side of the correlation function contains contributions from  $1S$ ,  $1P$  and  $2S$  states at the same time. However, it is impossible to analytically solve the resultant coupled equations and separate different the contributions from each other when three resonances are involved. For this reason, in this work we use the ansatz that the hadronic side contains contributions from either  $1S + 1P$  or  $1S + 2S$  states. In this way, we assume that the observed states  $\Sigma_b(6097)^\pm$  are either  $1P$  or  $2S$  excitations of the corresponding ground-state baryons with  $J = \frac{1}{2}$  or  $J = \frac{3}{2}$ . Then we separate the corresponding contributions of each state in each case. Naturally, such an assumption brings some systematic uncertainties. However, in order to estimate the order of the systematic uncertainties due to this assumption it is also necessary to simultaneously

take into account the contributions from the  $1P$  and  $2S$  states. In this case, we need to numerically solve the resultant three coupled equations. An analysis of this scenario lies beyond the scope of this work, and we plan on discussing this point separately in the future.

We use the matrix elements given by Eq. (1) in Eqs. (6) and (7), together with the following matrix elements defined in terms of the decay constants,  $\lambda^{(*)}$ ,  $\lambda_1^{(*)}$ ,  $\lambda_2^{(*)}$ , and  $\lambda_{\Lambda_b}$ :

$$\begin{aligned} \langle 0 | \eta_{\Sigma_b} | \Sigma_b(p', s) \rangle &= \lambda u(p', s), \\ \langle 0 | \eta_{\Sigma_b} | \Sigma_{b1}(p', s) \rangle &= \lambda_1 \gamma_5 u(p', s), \\ \langle 0 | \eta_{\Sigma_b} | \Sigma_{b2}(p', s) \rangle &= \lambda_2 u(p', s), \\ \langle 0 | \eta_{\Lambda_b} | \Lambda_b(p, s) \rangle &= \lambda_{\Lambda_b} u(p, s) \end{aligned} \quad (8)$$

for the  $J = \frac{1}{2}$  states and

$$\begin{aligned} \langle 0 | \eta_{\Sigma_b^* \mu} | \Sigma_b^*(p', s) \rangle &= \lambda^* u_\mu(p', s), \\ \langle 0 | \eta_{\Sigma_b^* \mu} | \Sigma_{b1}^*(p', s) \rangle &= \lambda_1^* \gamma_5 u_\mu(p', s), \\ \langle 0 | \eta_{\Sigma_b^* \mu} | \Sigma_{b2}^*(p', s) \rangle &= \lambda_2^* u_\mu(p', s) \end{aligned} \quad (9)$$

for the  $J = \frac{3}{2}$  states. We perform the summations over spins using

$$\sum_s u(k, s) \bar{u}(k, s) = (\not{k} + m), \quad (10)$$

$$\sum_s u_\mu(k, s) \bar{u}_\nu(k, s) = -(\not{k} + m) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{2k_\mu k_\nu}{3m^2} + \frac{k_\mu \gamma_\nu - k_\nu \gamma_\mu}{3m} \right], \quad (11)$$

and thus we obtain

$$\Pi^{\text{Phys}}(p, q) = \frac{g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m^2)} (\not{q} \not{p} \gamma_5 + (m - m_{\Lambda_b}) \not{p} \gamma_5) + \frac{g_{\Sigma_{b1} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m_1^2)} (\not{q} \not{p} \gamma_5 - (m_1 + m_{\Lambda_b}) \not{p} \gamma_5) + \dots, \quad (12)$$

$$\Pi^{\text{Phys}}(p, q) = \frac{g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m^2)} (\not{q} \not{p} \gamma_5 + (m - m_{\Lambda_b}) \not{p} \gamma_5) + \frac{g_{\Sigma_{b2} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m_2^2)} (\not{q} \not{p} \gamma_5 + (m_2 - m_{\Lambda_b}) \not{p} \gamma_5) + \dots, \quad (13)$$

$$\begin{aligned}
\Pi_{\mu}^{\text{Phys}}(p, q) = & -\frac{g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m^*)} \left[ \frac{(m_{\Lambda_b}^2 + 2m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2)}{6m^*} \not{q} \not{p} \gamma_{\mu} + \frac{(m_{\Lambda_b}^2 - m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m^{*2}} \not{q} q_{\mu} \right] \\
& + \frac{g_{\Sigma_{b1}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1^*}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m_1^{*2})} \left[ \frac{(m_{\Lambda_b}^2 - 2m_{\Lambda_b} m_1^* + m_1^{*2} - m_{\pi}^2)}{6m_1^*} \not{q} \not{p} \gamma_{\mu} - \frac{(m_{\Lambda_b}^2 + m_{\Lambda_b} m_1^* + m_1^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m_1^{*2}} \not{q} q_{\mu} \right] \\
& + \dots, \tag{14}
\end{aligned}$$

$$\begin{aligned}
\Pi_{\mu}^{\text{Phys}}(p, q) = & -\frac{g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m^*)} \left[ \frac{(m_{\Lambda_b}^2 + 2m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2)}{6m^*} \not{q} \not{p} \gamma_{\mu} + \frac{(m_{\Lambda_b}^2 - m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m^{*2}} \not{q} q_{\mu} \right] \\
& - \frac{g_{\Sigma_{b2}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2^*}{(p^2 - m_{\Lambda_b}^2)(p'^2 - m_2^{*2})} \left[ \frac{(m_{\Lambda_b}^2 + 2m_{\Lambda_b} m_2^* + m_2^{*2} - m_{\pi}^2)}{6m_2^*} \not{q} \not{p} \gamma_{\mu} + \frac{(m_{\Lambda_b}^2 - m_{\Lambda_b} m_2^* + m_2^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m_2^{*2}} \not{q} q_{\mu} \right] \\
& + \dots, \tag{15}
\end{aligned}$$

where we only keep the terms that we use in the analyses and the dots in all of the final results represent contributions coming from other structures as well as the higher states and the continuum. By performing a double Borel transformation with respect to  $-p^2$  and  $-p'^2$  we suppress the contributions of the higher states and the continuum, and after this process Eqs. (12)–(15) become

$$\begin{aligned}
\tilde{\Pi}^{\text{Phys}}(p, q) = & g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda e^{-m^2/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} [\not{q} \not{p} \gamma_5 + (m - m_{\Lambda_b}) \not{p} \gamma_5] + g_{\Sigma_{b1} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1 e^{-m_1^2/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} \\
& \times [\not{q} \not{p} \gamma_5 - (m_1 + m_{\Lambda_b}) \not{p} \gamma_5] + \dots, \tag{16}
\end{aligned}$$

$$\begin{aligned}
\tilde{\Pi}^{\text{Phys}}(p, q) = & g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda e^{-m^2/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} [\not{q} \not{p} \gamma_5 + (m - m_{\Lambda_b}) \not{p} \gamma_5] + g_{\Sigma_{b2} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2 e^{-m_2^2/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} \\
& \times [\not{q} \not{p} \gamma_5 + (m_2 - m_{\Lambda_b}) \not{p} \gamma_5] + \dots, \tag{17}
\end{aligned}$$

$$\begin{aligned}
\tilde{\Pi}_{\mu}^{\text{Phys}}(p, q) = & -g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda^* e^{-m^{*2}/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} \left[ \frac{(m_{\Lambda_b}^2 + 2m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2)}{6m^*} \not{q} \not{p} \gamma_{\mu} \right. \\
& \left. + \frac{(m_{\Lambda_b}^2 - m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m^{*2}} \not{q} q_{\mu} \right] + g_{\Sigma_{b1}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1^* e^{-m_1^{*2}/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} \\
& \times \left[ \frac{(m_{\Lambda_b}^2 - 2m_{\Lambda_b} m_1^* + m_1^{*2} - m_{\pi}^2)}{6m_1^*} \not{q} \not{p} \gamma_{\mu} - \frac{(m_{\Lambda_b}^2 + m_{\Lambda_b} m_1^* + m_1^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m_1^{*2}} \not{q} q_{\mu} \right] \\
& + \dots, \tag{18}
\end{aligned}$$

$$\begin{aligned}
\tilde{\Pi}_{\mu}^{\text{Phys}}(p, q) = & -g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda^* e^{-m^{*2}/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} \left[ \frac{(m_{\Lambda_b}^2 + 2m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2)}{6m^*} \not{q} \not{p} \gamma_{\mu} \right. \\
& \left. + \frac{(m_{\Lambda_b}^2 - m_{\Lambda_b} m^* + m^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m^{*2}} \not{q} q_{\mu} \right] - g_{\Sigma_{b2}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2^* e^{-m_2^{*2}/M_1^2} e^{-m_{\Lambda_b}^2/M_2^2} \\
& \times \left[ \frac{(m_{\Lambda_b}^2 + 2m_{\Lambda_b} m_2^* + m_2^{*2} - m_{\pi}^2)}{6m_2^*} \not{q} \not{p} \gamma_{\mu} + \frac{(m_{\Lambda_b}^2 - m_{\Lambda_b} m_2^* + m_2^{*2} - m_{\pi}^2) m_{\Lambda_b}}{3m_2^{*2}} \not{q} q_{\mu} \right] \\
& + \dots, \tag{19}
\end{aligned}$$

where  $M_1^2$  and  $M_2^2$  are the corresponding Borel parameters to be fixed later. In the above equations, the notation  $\tilde{\Pi}_{(\mu)}^{\text{Phys}}(p, q)$  is used to show the Borel-transformed form of  $\Pi_{(\mu)}^{\text{Phys}}(p, q)$ , and we use  $q^2 = m_{\pi}^2$ . To get the sum rules for the coupling constants, we choose  $\not{q} \not{p} \gamma_5$  and  $\not{p} \gamma_5$  from the presented Lorentz structures for the  $J = \frac{1}{2}$  scenarios. The structures considered for the  $J = \frac{3}{2}$  scenarios are  $\not{q} \not{p} \gamma_{\mu}$  and  $\not{q} q_{\mu}$ . For the  $J = \frac{3}{2}$  scenarios, the selected structures are free of the undesired spin- $\frac{1}{2}$  contribution.

Besides the physical sides of the calculations we need the theoretical or QCD sides of the desired sum rules obtained from the correlation function (2) via the operator product expansion (OPE). To this end, the explicit forms of the interpolating

currents are placed in the correlator and possible contractions are made between the quark fields using Wick's theorem. As a result of these contractions, we obtain our results in terms of the heavy- and light-quark propagators. There also appear terms containing the matrix elements of the quark-gluon field operators between vacuum and  $\pi$ -meson states having the common form  $\langle \pi(q) | \bar{q}(x) \Gamma G_{\mu\nu} q(y) | 0 \rangle$  or  $\langle \pi(q) | \bar{q}(x) \Gamma q(y) | 0 \rangle$ . Their explicit expressions are given in terms of the  $\pi$ -meson distribution amplitudes (DAs) (see Refs. [18–20]).  $\Gamma$  and  $G_{\mu\nu}$  denote the full set of Dirac matrices and the gluon field-strength tensor, respectively. Using these matrix elements,

one gets the nonperturbative parts contributing to the results in coordinate space. We then carry out the calculations in momentum space and apply a double Borel transformation over the same variables as the physical sides. After applying the continuum subtraction procedure, the coefficients of same Lorentz structures as in the physical sides are considered, and the matching of these coefficients from both sides leads to the QCD sum rules for the strong coupling constants under question. Representing the Borel-transformed results of the QCD sides with  $\tilde{\Pi}_1^{(*)\text{OPE}}$  and  $\tilde{\Pi}_2^{(*)\text{OPE}}$ , we can depict the above-mentioned matches as follows:

$$g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda e^{-\frac{m^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} + g_{\Sigma_{b1} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1 e^{-\frac{m_1^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} = \tilde{\Pi}_1^{\text{OPE}},$$

$$g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda e^{-\frac{m^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} (m - m_{\Lambda_b}) - g_{\Sigma_{b1} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1 e^{-\frac{m_1^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} (m_1 + m_{\Lambda_b}) = \tilde{\Pi}_2^{\text{OPE}}, \quad (20)$$

$$g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda e^{-\frac{m^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} + g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1 e^{-\frac{m_1^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} = \tilde{\Pi}_1^{\text{OPE}},$$

$$g_{\Sigma_b \Lambda_b \pi} \lambda_{\Lambda_b} \lambda e^{-\frac{m^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} (m - m_{\Lambda_b}) + g_{\Sigma_{b2} \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2 e^{-\frac{m_2^2}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} (m_2 - m_{\Lambda_b}) = \tilde{\Pi}_2^{\text{OPE}}, \quad (21)$$

$$- g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda^* \frac{[(m^* + m_{\Lambda_b})^2 - m_\pi^2]}{6m^*} e^{-\frac{m^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} + g_{\Sigma_{b1}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1^* \frac{[m_1^* - m_{\Lambda_b}]^2 - m_\pi^2]}{6m_1^*} e^{-\frac{m_1^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} = \tilde{\Pi}_1^{*\text{OPE}},$$

$$- g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda^* \frac{[m^{*2} + m_{\Lambda_b}^2 - m^* m_{\Lambda_b} - m_\pi^2] m_{\Lambda_b}}{3m^{*2}} e^{-\frac{m^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} - g_{\Sigma_{b1}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_1^* \frac{[m_1^{*2} + m_{\Lambda_b}^2 + m_1^* m_{\Lambda_b} - m_\pi^2] m_{\Lambda_b}}{3m_1^{*2}} e^{-\frac{m_1^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} = \tilde{\Pi}_2^{*\text{OPE}}, \quad (22)$$

$$- g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda^* \frac{[(m^* + m_{\Lambda_b})^2 - m_\pi^2]}{6m^*} e^{-\frac{m^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} - g_{\Sigma_{b2}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2^* \frac{[(m_2^* + m_{\Lambda_b})^2 - m_\pi^2]}{6m_2^*} e^{-\frac{m_2^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} = \tilde{\Pi}_1^{*\text{OPE}},$$

$$- g_{\Sigma_b^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda^* \frac{[m^{*2} + m_{\Lambda_b}^2 - m^* m_{\Lambda_b} - m_\pi^2] m_{\Lambda_b}}{3m^{*2}} e^{-\frac{m^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} - g_{\Sigma_{b2}^* \Lambda_b \pi} \lambda_{\Lambda_b} \lambda_2^* \frac{[m_2^{*2} + m_{\Lambda_b}^2 - m_2^* m_{\Lambda_b} - m_\pi^2] m_{\Lambda_b}}{3m_2^{*2}} e^{-\frac{m_2^{*2}}{M_1^2}} e^{-\frac{m_{\Lambda_b}}{M_2^2}} = \tilde{\Pi}_2^{*\text{OPE}}, \quad (23)$$

where  $\tilde{\Pi}_1^{\text{OPE}}$  ( $\tilde{\Pi}_1^{*\text{OPE}}$ ) and  $\tilde{\Pi}_2^{\text{OPE}}$  ( $\tilde{\Pi}_2^{*\text{OPE}}$ ) represent the Borel-transformed coefficients of the  $\not{q} \not{p} \gamma_5$  ( $\not{q} \not{p} \gamma_\mu$ ) and  $\not{p} \gamma_5$  ( $\not{q} q_\mu$ ) structures for the  $J = \frac{1}{2}$  ( $\frac{3}{2}$ ) cases. The procedures for calculating these functions and their expressions are very lengthy. Hence, in the Appendix we briefly show how we calculated these functions and give only the explicit form of the  $\tilde{\Pi}_1^{*\text{OPE}}$  function for the  $\Sigma_b(6097)^+ \rightarrow \Lambda_b^0 \pi^+$  transition as an illustration.

The QCD sum rules for the coupling constants are obtained from the numerical solutions of the pairs of equations given in Eqs. (20) and (21) for the  $J = \frac{1}{2}$  scenarios and in Eqs. (22) and (23) for the  $J = \frac{3}{2}$  scenarios.

Calculating the coupling constants requires some input parameters, presented in Table I. Since the masses of the considered baryons are close to each other, we choose

$$M_1^2 = M_2^2 = 2M^2 \quad \text{obtained from} \quad M^2 = \frac{M_1^2 M_2^2}{M_1^2 + M_2^2}. \quad (24)$$

As is seen from Eqs. (20)–(23), in order to analyze the coupling constants we also need the masses and decay constants of the baryons. To obtain the masses and decay constants we consider the following correlation function:

TABLE I. Some input parameters used in the calculations of the coupling constants and the masses.

Parameters	Values
$m_{\Sigma_b^+}$	5811.31.9 MeV [21]
$m_{\Sigma_b^-}$	$5815.5 \pm 1.8$ MeV [21]
$m_{\Sigma_b^{*+}}$	$5832.1 \pm 1.9$ MeV [21]
$m_{\Sigma_b^{*-}}$	$5835.1 \pm 1.9$ MeV [21]
$m_b$	$4.18_{-0.03}^{+0.04}$ GeV [21]
$m_d$	$4.7_{-0.3}^{+0.5}$ MeV [21]
$\lambda_{\Lambda_b}$	$(3.85 \pm 0.56) \times 10^{-2}$ GeV <sup>3</sup> [16]
$\langle \bar{q}q \rangle (1 \text{ GeV})$	$(-0.24 \pm 0.01)^3$ GeV <sup>3</sup> [22]
$\langle \bar{s}s \rangle$	$0.8 \langle \bar{q}q \rangle$ [22]
$m_0^2$	$(0.8 \pm 0.1)$ GeV <sup>2</sup> [22]
$\langle g_s^2 G^2 \rangle$	$4\pi^2 (0.012 \pm 0.004)$ GeV <sup>4</sup> [23]

$$T_{(\mu\nu)}(q) = i \int d^4x e^{iq \cdot x} \langle 0 | \mathcal{T} \{ \eta_{\Sigma_b^{(*)}(\mu)}(x) \bar{\eta}_{\Sigma_b^{(*)}(\nu)}(0) \} | 0 \rangle, \quad (25)$$

where the current  $\eta_{\Sigma_b^{(*)}(\mu)}$  corresponds to the considered  $J = \frac{1}{2}(\frac{3}{2})$  state, composed of the quark fields with the related quantum numbers. The subindex  $\Sigma_b$  is used to represent one of the states:  $\Sigma_b^\pm$  with spin  $\frac{1}{2}$  or  $\Sigma_b^{*\pm}$  with  $J = \frac{3}{2}$ . To determine the masses of the  $\Sigma_b^{(*)}$  states, we again consider two assumptions for each of the above-mentioned baryons, and four different QCD sum rules are obtained. For this purpose, the interpolating currents given in Eqs. (4) and (5) are used.

In the two-point QCD sum rule method for mass, one uses two methods to calculate the corresponding correlator. The first one includes the calculation of the correlator in terms of the hadronic degrees of freedom, and therefore it is called the physical or phenomenological side. For this purpose, the interpolating fields are treated as the operators creating or annihilating the states under consideration. Insertion of complete sets of hadronic states having the same quantum numbers of the hadrons under question results in

$$T_{\mu\nu}^{\text{Phys}}(q) = -\frac{\lambda^{*2}}{q^2 - m^{*2}} (\not{q} + m^*) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{2q_\mu q_\nu}{3m^{*2}} + \frac{q_\mu \gamma_\nu - q_\nu \gamma_\mu}{3m^*} \right] - \frac{\lambda_1^{*2}}{q^2 - m_1^{*2}} (\not{q} - m_1^*) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{2q_\mu q_\nu}{3m_1^{*2}} + \frac{q_\mu \gamma_\nu - q_\nu \gamma_\mu}{3m_1^*} \right] + \dots, \quad (30)$$

and

$$T_{\mu\nu}^{\text{Phys}}(q) = -\frac{\lambda^{*2}}{q^2 - m^{*2}} (\not{q} + m^*) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{2q_\mu q_\nu}{3m^{*2}} + \frac{q_\mu \gamma_\nu - q_\nu \gamma_\mu}{3m^*} \right] - \frac{\lambda_2^{*2}}{q^2 - m_2^{*2}} (\not{q} + m_2^*) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{2q_\mu q_\nu}{3m_2^{*2}} + \frac{q_\mu \gamma_\nu - q_\nu \gamma_\mu}{3m_2^*} \right] + \dots \quad (31)$$

$$T_{(\mu\nu)}^{\text{Phys}}(q) = \frac{\langle 0 | \eta_{\Sigma_b^{(*)}(\mu)} | \Sigma_b^{(*)}(q, s) \rangle \langle \Sigma_b^{(*)}(q, s) | \bar{\eta}_{\Sigma_b^{(*)}(\nu)} | 0 \rangle}{m^{(*)2} - q^2} + \frac{\langle 0 | \eta_{\Sigma_b^{(*)}(\mu)} | \Sigma_{b1}^{(*)}(q, s) \rangle \langle \Sigma_{b1}^{(*)}(q, s) | \bar{\eta}_{\Sigma_b^{(*)}(\nu)} | 0 \rangle}{m_1^{(*)2} - q^2} + \dots, \quad (26)$$

and

$$T_{(\mu\nu)}^{\text{Phys}}(q) = \frac{\langle 0 | \eta_{\Sigma_b^{(*)}(\mu)} | \Sigma_b^{(*)}(q, s) \rangle \langle \Sigma_b^{(*)}(q, s) | \bar{\eta}_{\Sigma_b^{(*)}(\nu)} | 0 \rangle}{m^{(*)2} - q^2} + \frac{\langle 0 | \eta_{\Sigma_b^{(*)}(\mu)} | \Sigma_{b2}^{(*)}(q, s) \rangle \langle \Sigma_{b2}^{(*)}(q, s) | \bar{\eta}_{\Sigma_b^{(*)}(\nu)} | 0 \rangle}{m_2^{(*)2} - q^2} + \dots, \quad (27)$$

where Eqs. (26) and (27) are obtained for the  $1P$  and  $2S$  excitation scenarios, respectively, and  $m^{(*)}$ ,  $m_1^{(*)}$ , and  $m_2^{(*)}$  are the masses of the  $1S$ ,  $1P$ , and  $2S$  excited states of each considered  $\Sigma_b^{(*)}$  baryon whose one-particle states are represented by  $|\Sigma_b^{(*)}\rangle$ ,  $|\Sigma_{b1}^{(*)}\rangle$ , and  $|\Sigma_{b2}^{(*)}\rangle$ , respectively. The dots represent contributions of the higher states and the continuum. As can be seen from the above equations, these calculations also require the matrix elements given in Eqs. (8) and (9). In these calculations, the ground state and its  $1P$  and  $2S$  excitations are again denoted by  $\Sigma_b(\Sigma_b^*)$ ,  $\Sigma_{b1}(\Sigma_{b1}^*)$ , and  $\Sigma_{b2}(\Sigma_{b2}^*)$  for the corresponding  $J = \frac{1}{2}(\frac{3}{2})$  baryons, respectively, and  $\lambda(\lambda^*)$ ,  $\lambda_1(\lambda_1^*)$ , and  $\lambda_2(\lambda_2^*)$  are their corresponding decay constants. After using the expressions for the matrix elements and the summation relations for spinors  $u(q, s)$  and  $u_\mu(q, s)$  given in Eqs (10) and (11), the physical sides for the  $J = \frac{1}{2}$  cases are obtained as

$$T^{\text{Phys}}(q) = \frac{\lambda^2(\not{q} + m)}{m^2 - q^2} + \frac{\lambda_1^2(\not{q} - m_1)}{m_1^2 - q^2} + \dots, \quad (28)$$

and

$$T^{\text{Phys}}(q) = \frac{\lambda^2(\not{q} + m)}{m^2 - q^2} + \frac{\lambda_2^2(\not{q} + m_2)}{m_2^2 - q^2} + \dots \quad (29)$$

Similar steps give the results for the  $J = \frac{3}{2}$  cases as

As already mentioned, we need to use a second method to calculate the same correlation function, Eq. (25), which proceeds in terms of the quark and gluon degrees of freedom. For this side of the calculation, we exploit the explicit expressions of the interpolating currents and OPE. After making the possible contractions between the quark fields, the results turn into expressions containing heavy- and light-quark propagators. To obtain the final results, the expressions of these quark propagators are used and a Fourier transformation from coordinate space to momentum space is performed to obtain the final form of the QCD sides. The results of this side are very lengthy; therefore, we will not give them here explicitly.

The calculations of the physical and QCD sides are followed by the application of a Borel transformation to both sides, which suppresses the contributions coming from the higher states and the continuum. Finally, the QCD sum rules are obtained by matching the coefficients of the same Lorentz structures from both sides. In the present work, the mentioned structures are  $\not{q}$  and  $I$  for the  $J = \frac{1}{2}$  cases and  $\not{q}g_{\mu\nu}$  and  $g_{\mu\nu}$  for the  $J = \frac{3}{2}$  cases. While choosing the structures for the  $J = \frac{3}{2}$  states, among the various possibilities, the structures  $\not{q}g_{\mu\nu}$  and  $g_{\mu\nu}$  are considered since the others contain undesired contributions from the  $J = \frac{1}{2}$  states as well. After applying continuum subtraction, the obtained equation pairs are solved numerically for each state under consideration. These equations are given as

$$\begin{aligned} \lambda^2 e^{-\frac{m^2}{M^2}} + \lambda_1^2 (\lambda_2^2) e^{-\frac{m_1^2(m_2^2)}{M^2}} &= \tilde{T}_1^{\text{OPE}}, \\ m\lambda^2 e^{-\frac{m^2}{M^2}} \mp m_1(m_2)\lambda_1^2 (\lambda_2^2) e^{-\frac{m_1^2(m_2^2)}{M^2}} &= \tilde{T}_2^{\text{OPE}}. \end{aligned} \quad (32)$$

In the second term of the second equation, we use the  $-$  and  $+$  signs to represent the results for the  $1P$  excitation ( $\Sigma_{b1}$ ) and  $2S$  excitation ( $\Sigma_{b2}$ ), respectively. To represent the expressions obtained in the QCD side of the calculations, we use  $\tilde{T}_i^{\text{OPE}}$  with  $i = 1, 2$ , which are the coefficients of the structures  $\not{q}$  and  $I$  for the  $J = \frac{1}{2}$  cases. To obtain the results corresponding to the  $J = \frac{3}{2}$  cases, it suffices to make the changes  $\lambda_1 \rightarrow \lambda_1^*$ ,  $\lambda_2 \rightarrow \lambda_2^*$ ,  $m_1 \rightarrow m_1^*$ ,  $m_2 \rightarrow m_2^*$ , and  $\tilde{T}_i^{\text{OPE}} \rightarrow \tilde{T}_i^{*\text{OPE}}$ , where  $\tilde{T}_i^{*\text{OPE}}$  is used to represent the coefficients obtained from  $\not{q}g_{\mu\nu}$  and  $g_{\mu\nu}$  in the QCD side.

In the numerical analyses of the obtained results, we need some input parameters, which are presented in Table I. The other ingredients of the sum rules are the three auxiliary parameters present in the results, namely, the Borel parameter  $M^2$ , the threshold parameter  $s_0$ , and an arbitrary parameter  $\beta$ . Note that the parameter  $\beta$  belongs to the currents of the states with  $J = \frac{1}{2}$ . Their working regions are fixed via following some criteria of the QCD sum rule formalism. To decide on the relevant region for the Borel parameter, the convergence of the OPE calculation is considered. To satisfy this requirement, we demand a

dominant perturbative contribution compared to the non-perturbative ones which helps us determine the lower limit of the Borel parameter. As for its upper limit, the criterion is pole dominance. Specifically, for the upper band of the Borel window we require that

$$\frac{\tilde{T}_i^{(*)\text{OPE}}(M^2, s_0, \beta)}{\tilde{T}_i^{(*)\text{OPE}}(M^2, \infty, \beta)} \geq \frac{1}{2}, \quad (33)$$

while for the lower band we require that the perturbative part in each case exceeds the total nonperturbative contributions and that the series of the corresponding OPE converge. From our analyses, we get the following working interval:

$$5 \text{ GeV}^2 \leq M^2 \leq 8 \text{ GeV}^2. \quad (34)$$

On the other hand, the threshold parameter  $s_0$  is related to the energy of the first excited state of the considered state. Due to the lack of information about these excited states, this parameter is also determined by using the pole-dominance condition, and we obtain

$$43 \text{ GeV}^2 \leq s_0 \leq 47 \text{ GeV}^2. \quad (35)$$

The parameter  $\beta$  is determined from the analyses of the results searching for the region giving the least possible variation with respect to this parameter. This region is acquired via a parametric plot depicting the dependency of the result on  $\cos \theta$ , where  $\beta = \tan \theta$ . As an example, in Fig. 1 we plot the dependence of the residue of the  $\Sigma_b^+(\frac{1}{2}^-)$  state on  $\cos \theta$  at average values of  $M^2$  and  $s_0$ . From this figure and the analyses of the obtained sum rules, the working region for  $\cos \theta$  is obtained as

$$-1.0 \leq \cos \theta \leq -0.3 \quad \text{and} \quad 0.3 \leq \cos \theta \leq 1.0, \quad (36)$$

where the results have small dependencies on the mixing parameter  $\beta$ . In order to see how the OPE sides of the mass

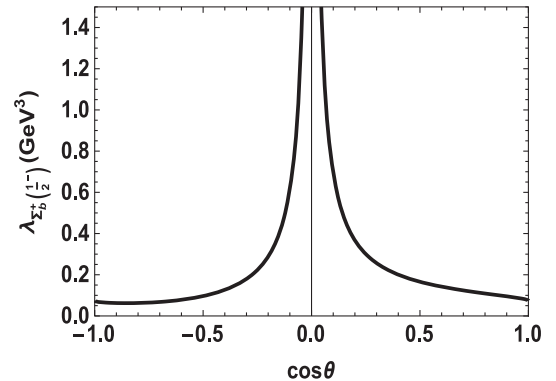


FIG. 1. The dependence of the residue of the  $\Sigma_b^+(\frac{1}{2}^-)$  state on  $\cos \theta$  at average values of  $M^2$  and  $s_0$ .

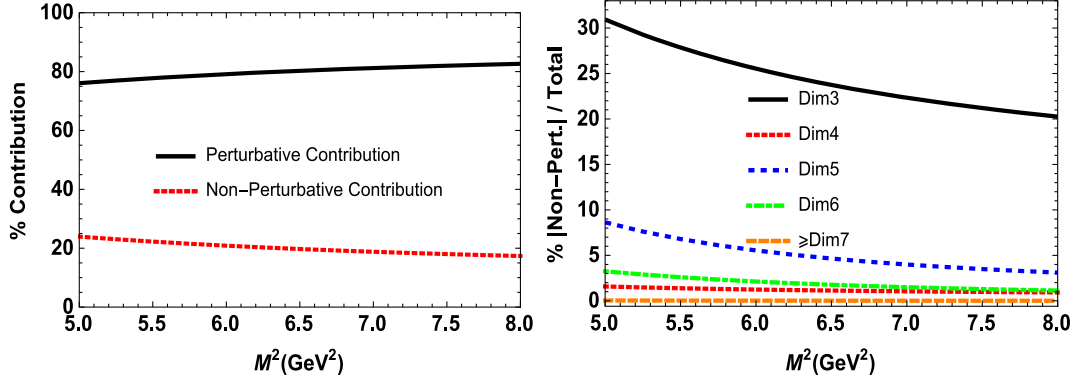


FIG. 2. Various contributions to the OPE side of the mass sum rules for the  $J = \frac{3}{2}$  case and the structure  $\not{q}g_{\mu\nu}$  on  $M^2$  at average values of  $s_0$  and  $\cos\theta$ .

sum rules converge, as an example we show the dependence of the OPE side of the mass sum rule for the  $J = \frac{3}{2}$  case and the structure  $\not{q}g_{\mu\nu}$  on  $M^2$  at average values of  $s_0$  and  $\cos\theta$  in Fig. 2. As is seen from this figure, the perturbative part constitutes the main contribution and the corresponding OPE series demonstrate a good convergence.

Given the working intervals of the auxiliary parameters and those given in Table I, the obtained masses and decay constants are presented in Table II. To extract the masses of the considered excited states, the masses of corresponding ground-state baryons are used as inputs. Note that the central values presented in this table are obtained at average values of  $M^2$  and  $s_0$ , i.e.,  $M^2 = 6.5 \text{ GeV}^2$  and  $s_0 = 45 \text{ GeV}^2$ , as well as average values of  $\cos\theta$  on both the positive and negative sides. This table also contains the errors in the results coming from uncertainties that exist in the input parameters and uncertainties arising from the determination of the working windows for the auxiliary parameters.

As seen from the table, although the masses are consistent with the experimental values  $m_{\Sigma_b(6097)^-} = 6098.0 \pm 1.7 \pm 0.5 \text{ MeV}$  and  $m_{\Sigma_b(6097)^+} = 6095.8 \pm 1.7 \pm 0.4 \text{ MeV}$

TABLE II. The results of the spectroscopic parameters obtained for the  $1P$  and  $2S$  excitations of the ground-state  $\Sigma_b^+$  and  $\Sigma_b^-$  baryons with  $J = \frac{1}{2}$  and  $\Sigma_b^{*+}$  and  $\Sigma_b^{*-}$  with  $J^P = \frac{3}{2}$ .

State	Mass (MeV)	Decay constant $\lambda(\text{GeV}^3)$
$\Sigma_b^+(\frac{1}{2}^-)(1P)$	$6091_{-168}^{+197}$	$0.11_{-0.03}^{+0.03}$
$\Sigma_b^+(\frac{1}{2}^+)(2S)$	$6091_{-168}^{+197}$	$0.73_{-0.04}^{+0.02}$
$\Sigma_b^-(\frac{1}{2}^-)(1P)$	$6092_{-168}^{+197}$	$0.11_{-0.03}^{+0.06}$
$\Sigma_b^-(\frac{1}{2}^+)(2S)$	$6092_{-168}^{+197}$	$0.74_{-0.02}^{+0.04}$
$\Sigma_b^{*+}(\frac{3}{2}^-)(1P)$	$6093_{-123}^{+108}$	$0.068_{-0.011}^{+0.010}$
$\Sigma_b^{*+}(\frac{3}{2}^+)(2S)$	$6093_{-123}^{+108}$	$0.47_{-0.02}^{+0.06}$
$\Sigma_b^{*-}(\frac{3}{2}^-)(1P)$	$6095_{-122}^{+107}$	$0.068_{-0.011}^{+0.010}$
$\Sigma_b^{*-}(\frac{3}{2}^+)(2S)$	$6095_{-122}^{+107}$	$0.47_{-0.02}^{+0.04}$

[7], their central values are too close to accurately predict the quantum numbers of the observed  $\Sigma_b(6097)$  states. Therefore, for this purpose it would be much more helpful to resort to the results obtained for the decay widths. These decay widths are obtained from the results of the strong coupling constant calculations and the obtained mass and decay constant values.

After obtaining the masses and decay constants, we turn our attention again to the strong coupling constant calculations in which the values of the above spectroscopic parameters are used as inputs. In the strong coupling constant analyses we adopt the auxiliary parameters used in the calculations of masses and decay constants with one exception. The Borel parameter  $M^2$  in these calculations is revisited, and, considering the OPE series convergence and the pole dominance conditions, its interval for the strong coupling constants is determined as

$$15 \text{ GeV}^2 \leq M^2 \leq 25 \text{ GeV}^2. \quad (37)$$

The coupling constants obtained from the QCD sum rule analyses are used to get the related decay widths for the  $1P$  and the  $2S$  excitations of the considered states. To this end, we use the decay width formulas for the  $J = \frac{1}{2}$  cases, which are

$$\Gamma(\Sigma_{b1} \rightarrow \Lambda_b \pi) = \frac{g_{\Sigma_{b1} \Lambda_b \pi}^2}{8\pi m_1^2} [(m_1 + m_{\Lambda_b})^2 - m_\pi^2] f(m_1, m_{\Lambda_b}, m_\pi) \quad (38)$$

for the  $1P$  excitations and

$$\Gamma(\Sigma_{b2} \rightarrow \Lambda_b \pi) = \frac{g_{\Sigma_{b2} \Lambda_b \pi}^2}{8\pi m_2^2} [(m_2 - m_{\Lambda_b})^2 - m_\pi^2] f(m_2, m_{\Lambda_b}, m_\pi) \quad (39)$$

for the  $2S$  excitations. For the  $J = \frac{3}{2}$  cases the respective decay-width equations are



TABLE III. The results for the coupling constants and decay widths obtained for  $1P$  and  $2S$  excitations of the ground-state  $\Sigma_b^+$  and  $\Sigma_b^-$  baryons with spin  $\frac{1}{2}$  and  $\Sigma_b^{*+}$  and  $\Sigma_b^{*-}$  with spin  $\frac{3}{2}$ .

State $B(J^P)$	$g_{\Sigma_b \Lambda_b \pi}$	$\Gamma$ (MeV)
$\Sigma_b^+(\frac{1}{2}^-)(1P)$	$1.4 \pm 0.3$	$127.2 \pm 36.9$
$\Sigma_b^+(\frac{1}{2}^+)(2S)$	$9.1 \pm 2.0$	$7.7 \pm 2.3$
$\Sigma_b^-(\frac{1}{2}^-)(1P)$	$1.2 \pm 0.3$	$85.4 \pm 23.1$
$\Sigma_b^-(\frac{1}{2}^+)(2S)$	$7.5 \pm 1.7$	$5.4 \pm 1.6$
$\Sigma_b^{*+}(\frac{3}{2}^-)(1P)$	$67.7 \pm 14.9$	$27.5 \pm 7.4$
$\Sigma_b^{*+}(\frac{3}{2}^+)(2S)$	$37.8 \pm 8.3$	$5.7 \pm 1.6$
$\Sigma_b^{*-}(\frac{3}{2}^-)(1P)$	$67.7 \pm 14.9$	$28.1 \pm 7.6$
$\Sigma_b^{*-}(\frac{3}{2}^+)(2S)$	$37.8 \pm 8.3$	$5.8 \pm 1.6$

$$\Gamma(\Sigma_{b1}^* \rightarrow \Lambda_b \pi) = \frac{g_{\Sigma_{b1}^* \Lambda_b \pi}^2}{24\pi m_1^*} [(m_1^* - m_{\Lambda_b})^2 - m_\pi^2] f^3(m_1^*, m_{\Lambda_b}, m_\pi) \quad (40)$$

and

$$\Gamma(\Sigma_{b2}^* \rightarrow \Lambda_b \pi) = \frac{g_{\Sigma_{b2}^* \Lambda_b \pi}^2}{24\pi m_2^{*2}} [(m_2^* + m_{\Lambda_b})^2 - m_\pi^2] f^3(m_2^*, m_{\Lambda_b}, m_\pi). \quad (41)$$

The function  $f(x, y, z)$  in the decay width equations is

$$f(x, y, z) = \frac{1}{2x} \sqrt{x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2}.$$

Table III presents the numerical results of the calculations for the coupling constants and decay widths. It can be seen from the table that our width results obtained for the scenario where  $\Sigma_b(6097)^\pm$  are the  $1P$  excitations of the ground-state  $\Sigma_b^{*\pm}$  with  $J^P = \frac{3}{2}^-$  are comparable to the experimental values  $\Gamma_{\Sigma_b(6097)^-} = 28.9 \pm 4.2 \pm 0.9$  and

$\Gamma_{\Sigma_b(6097)^+} = 31.0 \pm 5.5 \pm 0.7$  MeV [7]. Note that the main uncertainties in the results for the couplings and masses belong to the variations of the results with respect to the variations of the continuum threshold  $s_0$  and the results show small dependencies on other auxiliary parameters as well as other input parameters. Figure 3 shows the dependence of  $g_{\Sigma_b^{*+}(\frac{3}{2}^-)\Lambda_b\pi}$  on  $M^2$  ( $s_0$ ) at different fixed values of  $s_0$  ( $M^2$ ) and at average values of  $\cos\theta$ . As is seen, the main source of uncertainties is the variation of the continuum threshold  $s_0$ .

To end this section, we will compare our results for the masses and widths with the predictions of other approaches. Using the quasi-two-body method, Ref. [8] obtained 6094 and 6098 MeV for the masses of the  $\Sigma(3/2^-)$  and  $\Sigma(5/2^-)$  states, respectively, indicating the possibility that the particle  $\Sigma_b(6097)$  has either  $J^P = 3/2^-$  or  $5/2^-$ . This result for the mass of the  $\Sigma(3/2^-)$  state is consistent with our predictions. In the same paper, the decay widths were also considered, and for the channel with final states  $\Lambda_b\pi$  they obtained 35.2 and 35.8 MeV for the  $\Sigma(3/2^-)$  and  $\Sigma(5/2^-)$  states, respectively, supporting their conclusion obtained from the mass calculations. The decay width to the same final state for the strong decay of the P-wave  $\Sigma_b$  baryon was also considered in Ref. [9] using the chiral quark model, yielding 32.3 and 31.4 MeV for the  $J^P = 3/2^-$  and  $5/2^-$  cases, respectively. Another study supporting the idea that the  $\Sigma_b(6097)$  state has either  $J^P = 3/2^-$  or  $5/2^-$  calculated the decay widths to be 14.56 [14.19] MeV for the  $\Sigma_b(6097)^- [\Sigma_b(6097)^+]$  state for both the  $J^P = 3/2^-$  and  $5/2^-$  cases [10]. As is seen, the results of Ref. [10] for the decay widths differ from our predictions and the experimental data considerably. However, the predictions of Refs. [8,9] are close to our predictions as well as the experimental results. The advantage of our predictions for the widths using the LCSR is that by combining these predictions with the mass results we can exactly assign the particles  $\Sigma_b(6097)^\pm$  to be the  $1P$  excitations of the ground-state  $\Sigma_b^{*\pm}$  baryons with quantum numbers  $J^P = \frac{3}{2}^-$ .

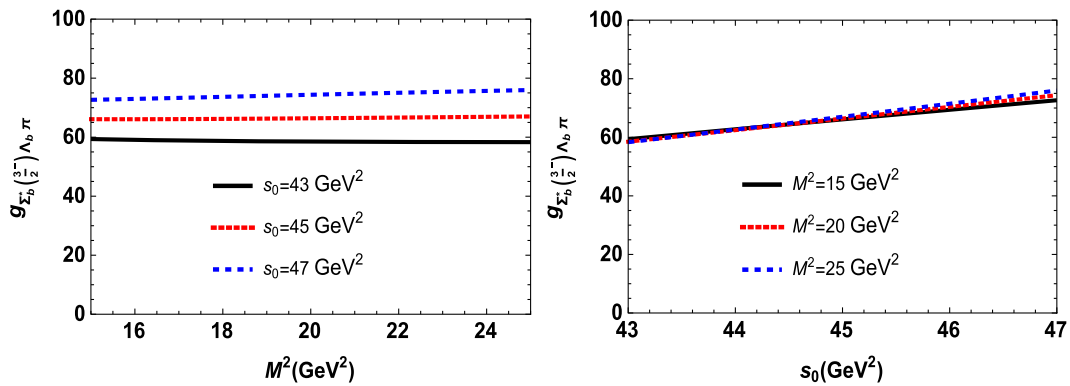


FIG. 3. The dependence of  $g_{\Sigma_b^{*+}(\frac{3}{2}^-)\Lambda_b\pi}$  on  $M^2$  ( $s_0$ ) at different fixed values of  $s_0$  ( $M^2$ ) and at average values of  $\cos\theta$ .

### III. CONCLUSION

To investigate the properties of the recently observed  $\Sigma_b(6097)^\pm$  states, we calculated the strong coupling constants for their transitions to  $\Lambda_b^0 \pi^\pm$  states using the light-cone QCD sum rules approach. We considered the two possible cases  $J = \frac{1}{2}$  and  $J = \frac{3}{2}$ , and both  $1P$  and  $2S$  excitations were taken into account. For each case, the considered decays were studied, and from the obtained strong coupling constants the related decay widths were calculated. To calculate the strong coupling constants, the mass and decay constant of each considered state for each possible quantum number were required. To obtain these quantities we employed the two-point QCD sum rules. We obtained the mass values  $m_{\Sigma_b^+(\frac{1}{2}; 1P(2S))} = 6091_{-168}^{+197}$ ,  $m_{\Sigma_b^-(\frac{1}{2}; 1P(2S))} = 6092_{-168}^{+197}$ ,  $m_{\Sigma_b^+(\frac{3}{2}; 1P(2S))} = 6093_{-123}^{+108}$ , and  $m_{\Sigma_b^-(\frac{3}{2}; 1P(2S))} = 6095_{-122}^{+107}$  MeV. As is seen, the central values obtained for the masses are consistent with the experimentally observed masses  $m(\Sigma_b(6097)^-) = 6098.0 \pm 1.7 \pm 0.5$  and  $m(\Sigma_b(6097)^+) = 6095.8 \pm 1.7 \pm 0.4$  MeV [7]. However, it is clear from these results that it is not possible to draw a conclusion about the quantum numbers of the states  $\Sigma_b(6097)^\pm$ , as the central values of the obtained results are close not only to the experimental results but also to each other, and this does not allow us to make a conclusive statement about the quantum numbers. Therefore, using them as input quantities in the calculations of the strong coupling constants, we obtained the numerical values of the corresponding coupling constants and subsequently the related decay widths, which

were the main focus of the present work. Our results for the decay widths obtained for the  $J^P = \frac{3}{2}^-$  states are  $\Gamma_{\Sigma_b^+(\frac{3}{2}^-)} = 27.5 \pm 7.4$  and  $\Gamma_{\Sigma_b^-(\frac{3}{2}^-)} = 28.1 \pm 7.6$  MeV, which are in accord with the observed widths of these states, i.e.,  $\Gamma(\Sigma_b(6097)^-) = 28.9 \pm 4.2 \pm 0.9$  and  $\Gamma(\Sigma_b(6097)^+) = 31.0 \pm 5.5 \pm 0.7$  MeV [7]. These results support the claim that the states are the  $1P$  excitations of the ground-state  $\Sigma_b^{*\pm}$  with  $J = \frac{3}{2}$ .

### ACKNOWLEDGMENTS

H. S. thanks Kocaeli University for the partial financial support through the grant BAP 2018/070.

### APPENDIX: SOME DETAILS OF THE CALCULATIONS OF THE SPECTRAL DENSITIES FOR THE COUPLING CONSTANTS

Here we present some details of the calculations of the spectral densities used in the analyses of the strong coupling constants. After contracting the quark fields on the QCD side, there appears an expression in terms of the heavy- and light-quark propagators as well as the matrix elements of the quark-gluon field operators between vacuum and pseudoscalar meson states having the forms  $\langle PS(q) | \bar{q}(x) \Gamma G_{\mu\nu} q(y) | 0 \rangle$  and  $\langle PS(q) | \bar{q}(x) \Gamma q(y) | 0 \rangle$ . These matrix elements are given in terms of the pseudoscalar meson DAs (see Refs. [18–20]). For some details on the calculations of the spectral densities in QCD, we also refer the reader to Ref. [24].

For the light- and heavy-quark propagators we use

$$S_q(x) = \frac{i\not{x}}{2\pi^2 x^4} - \frac{m_q}{4\pi^2 x^2} - \frac{\langle \bar{q}q \rangle}{12} \left( 1 - i \frac{m_q}{4} \not{x} \right) - \frac{x^2}{192} m_0^2 \langle \bar{q}q \rangle \left( 1 - i \frac{m_q}{6} \not{x} \right) - ig_s \int_0^1 du \left[ \frac{\not{x}}{16\pi^2 x^2} G_{\mu\nu}(ux) \sigma_{\mu\nu} - \frac{i}{4\pi^2 x^2} ux^\mu G_{\mu\nu}(ux) \gamma^\nu - i \frac{m_q}{32\pi^2} G_{\mu\nu}(ux) \sigma^{\mu\nu} \left( \ln \left( \frac{-x^2 \Lambda^2}{4} \right) + 2\gamma_E \right) \right] \quad (A1)$$

and

$$S_Q(x) = \frac{m_Q^2}{4\pi^2} \frac{K_1(m_Q \sqrt{-x^2})}{\sqrt{-x^2}} - i \frac{m_Q^2 \not{x}}{4\pi^2 x^2} K_2(m_Q \sqrt{-x^2}) - ig_s \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \int_0^1 du \left[ \frac{\not{k} + m_Q}{2(m_Q^2 - k^2)^2} G^{\mu\nu}(ux) \sigma_{\mu\nu} + \frac{u}{m_Q^2 - k^2} x_\mu G^{\mu\nu}(ux) \gamma_\nu \right], \quad (A2)$$

where  $\gamma_E$  is the Euler constant,  $G_{\mu\nu}$  is the gluon field-strength tensor,  $\Lambda$  is the scale parameter, and  $K_\nu$  in the heavy propagator denotes the Bessel functions of the second kind.

After inserting the light- and heavy-quark propagators as well as the DAs of the pseudoscalar mesons, we get the following generic term as an example for the leading twist (see also Ref. [15]):

$$T = \int d^4 x e^{ipx} \int_0^1 du e^{iuqx} f(u) \frac{K_\nu(m_Q \sqrt{-x^2})}{(\sqrt{-x^2})^\nu}, \quad (A3)$$

where  $f(u)$  denotes the leading DAs. We need to perform Fourier and Borel transformations as well as continuum subtraction on this expression. To this end, we use the integral representation of the modified Bessel function as

$$K_\nu(m_Q \sqrt{-x^2}) = \frac{\Gamma(\nu + 1/2)2^\nu}{\sqrt{\pi}m_Q^\nu} \int_0^\infty dt \cos(m_Q t) \frac{(\sqrt{-x^2})^\nu}{(t^2 - x^2)^{\nu+1/2}}, \quad (\text{A4})$$

which leads to

$$T = \int d^4x \int_0^1 du e^{iPx} f(u) \frac{\Gamma(\nu + 1/2)2^\nu}{\sqrt{\pi}m_Q^\nu} \int_0^\infty dt \cos(m_Q t) \frac{1}{(\sqrt{-x^2})^{n-\nu} (t^2 - x^2)^{\nu+1/2}}, \quad (\text{A5})$$

where  $P = p + uq$ . By transferring the calculations into Euclidean space and using the identity

$$\frac{1}{Z^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \alpha^{n-1} e^{-\alpha Z}, \quad (\text{A6})$$

we get

$$T = \frac{-i2^\nu}{\sqrt{\pi}m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int_0^1 du f(u) \int_0^\infty dt e^{im_Q t} \int_0^\infty dy y^{\frac{n-\nu}{2}-1} \int_0^\infty dv v^{\nu-\frac{1}{2}} e^{-vt^2} \int d^4\tilde{x} e^{-i\tilde{P}\tilde{x} - y\tilde{x}^2 - v\tilde{x}^2}, \quad (\text{A7})$$

where the  $\sim$  refers to the vectors in Euclidean space. After performing the resultant Gaussian integral over four- $\tilde{x}$ , we end up with

$$T = \frac{-i2^\nu \pi^2}{\sqrt{\pi}m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int_0^1 du f(u) \int_0^\infty dt e^{im_Q t} \int_0^\infty dy y^{\frac{n-\nu}{2}-1} \int_0^\infty dv v^{\nu-\frac{1}{2}} e^{-vt^2} \frac{e^{-\frac{\tilde{p}^2}{4(y+v)}}}{(y+v)^2}. \quad (\text{A8})$$

The next step is to perform the integration over  $t$ , which leads to

$$T = \frac{-i2^\nu \pi^2}{m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int_0^1 du f(u) \int_0^\infty dy y^{\frac{n-\nu}{2}-1} \int_0^\infty dv v^{\nu-1} e^{-\frac{m_Q^2}{4v}} \frac{e^{-\frac{\tilde{p}^2}{4(y+v)}}}{(y+v)^2}. \quad (\text{A9})$$

Let us define the following new variables:

$$\lambda = v + y, \quad \tau = \frac{y}{v + y}. \quad (\text{A10})$$

Applying this, we obtain

$$T = \frac{-i2^\nu \pi^2}{m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int_0^1 du f(u) \int d\lambda \int d\tau \lambda^{\frac{n+\nu}{2}-3} \tau^{\frac{n-\nu}{2}-1} (1-\tau)^{\nu-1} e^{-\frac{m_Q^2}{4\lambda(1-\tau)}} e^{-\frac{\tilde{p}^2}{4\lambda}}. \quad (\text{A11})$$

Now, we perform a double Borel transformation with respect to  $\tilde{p}^2$  and  $(\tilde{p} + \tilde{p})^2$  with the help of

$$\mathcal{B}(M^2) e^{-\alpha p^2} = \delta(1/M^2 - \alpha), \quad (\text{A12})$$

which leads to

$$\mathcal{B}(M_1^2) \mathcal{B}(M_2^2) T = \frac{-i2^\nu \pi^2}{m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int_0^1 du f(u) \int d\lambda \int d\tau \lambda^{\frac{n+\nu}{2}-3} \tau^{\frac{n-\nu}{2}-1} (1-\tau)^{\nu-1} e^{-\frac{m_Q^2}{4\lambda(1-\tau)}} e^{-\frac{u(u-1)\tilde{q}^2}{4\lambda}} \delta\left(\frac{1}{M_1^2} - \frac{u}{4\lambda}\right) \delta\left(\frac{1}{M_2^2} - \frac{1-u}{4\lambda}\right). \quad (\text{A13})$$

In this step the integrals over  $u$  and  $\lambda$  are performed. As a result, we get

$$\mathcal{B}(M_1^2)\mathcal{B}(M_2^2)T = \frac{-i2^\nu 4^2 \pi^2}{m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int d\tau f(u_0) \left(\frac{M^2}{4}\right)^{\frac{n+\nu}{2}} \tau^{\frac{n-\nu}{2}-1} (1-\tau)^{\nu-1} e^{-\frac{m_Q^2}{M^2(1-\tau)}} e^{\frac{q^2}{M_1^2+M_2^2}}, \quad (\text{A14})$$

where  $u_0 = \frac{M_2^2}{M_1^2+M_2^2}$  and  $M^2 = \frac{M_1^2 M_2^2}{M_1^2+M_2^2}$ . By making the replacement  $\tau = x^2$ , we obtain

$$\mathcal{B}(M_1^2)\mathcal{B}(M_2^2)T = \frac{-i2^{\nu+1} 4^2 \pi^2}{m_Q^\nu \Gamma(\frac{n-\nu}{2})} \int_0^1 dx f(u_0) \left(\frac{M^2}{4}\right)^{\frac{n+\nu}{2}} x^{n-\nu-1} (1-x^2)^{\nu-1} e^{-\frac{m_Q^2}{M^2(1-x^2)}} e^{\frac{q^2}{M_1^2+M_2^2}}. \quad (\text{A15})$$

The last step is to change the variable  $\eta = \frac{1}{1-x^2}$  and use  $q^2 = m_p^2$ , which leads to

$$\mathcal{B}(M_1^2)\mathcal{B}(M_2^2)T = \frac{-i2^{\nu+1} 4^2 \pi^2}{m_Q^\nu \Gamma(\frac{n-\nu}{2})} f(u_0) \left(\frac{M^2}{4}\right)^{\frac{n+\nu}{2}} e^{-\frac{m_p^2}{M_1^2+M_2^2}} \Psi\left(\alpha, \beta, \frac{m_Q^2}{M^2}\right), \quad (\text{A16})$$

with

$$\Psi\left(\alpha, \beta, \frac{m_Q^2}{M^2}\right) = \frac{1}{\Gamma(\alpha)} \int_1^\infty d\eta e^{-\eta \frac{m_Q^2}{M^2}} \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1}, \quad (\text{A17})$$

where  $\alpha = \frac{n-\nu}{2}$  and  $\beta = 1-\nu$ .

At this stage, we discuss how the contributions of the higher states and the continuum are subtracted. We consider the generic form

$$A = (M^2)^n f(u_0) \Psi\left(\alpha, \beta, \frac{m_Q^2}{M^2}\right). \quad (\text{A18})$$

We are going to find the spectral density corresponding to this generic term. As a first step, we expand  $f(u_0)$  as

$$f(u_0) = \Sigma a_k u_0^k, \quad (\text{A19})$$

which leads to

$$A = \left(\frac{M_1^2 M_2^2}{M_1^2 + M_2^2}\right)^n \Sigma a_k \left(\frac{M_2^2}{M_1^2 + M_2^2}\right)^k \frac{1}{\Gamma(\alpha)} \int_1^\infty d\eta e^{-\eta \frac{m_Q^2}{M^2}} \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1}. \quad (\text{A20})$$

Now we introduce the new variables  $\sigma_1 = \frac{1}{M_1^2}$  and  $\sigma_2 = \frac{1}{M_2^2}$ . As a result, we get

$$\begin{aligned} A &= \Sigma a_k \frac{\sigma_1^k}{(\sigma_1 + \sigma_2)^{n+k}} \frac{1}{\Gamma(\alpha)} \int_1^\infty d\eta e^{-\eta m_Q^2 (\sigma_1 + \sigma_2)} \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} \\ &= \Sigma a_k \frac{\sigma_1^k}{\Gamma(n+k)\Gamma(\alpha)} \int_1^\infty d\eta e^{-\eta m_Q^2 (\sigma_1 + \sigma_2)} \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} \int_0^\infty d\xi e^{-\xi(\sigma_1 + \sigma_2)} \xi^{n+k-1} \\ &= \Sigma a_k \frac{\sigma_1^k}{\Gamma(n+k)\Gamma(\alpha)} \int_1^\infty d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} \int_0^\infty d\xi \xi^{n+k-1} e^{-(\xi + \eta m_Q^2)(\sigma_1 + \sigma_2)} \\ &= \Sigma a_k \frac{(-1)^k}{\Gamma(n+k)\Gamma(\alpha)} \int_1^\infty d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} \int_0^\infty d\xi \xi^{n+k-1} \left(\left(\frac{d}{d\xi}\right)^k e^{-(\xi + \eta m_Q^2)\sigma_1}\right) e^{-(\xi + \eta m_Q^2)\sigma_2}. \end{aligned} \quad (\text{A21})$$

By performing a double Borel transformation with respect to  $\sigma_1 \rightarrow \frac{1}{s_1}$  and  $\sigma_2 \rightarrow \frac{1}{s_2}$ , we obtain the following double spectral density:

$$\rho(s_1, s_2) = \Sigma a_k \frac{(-1)^k}{\Gamma(n+k)\Gamma(\alpha)} \int_1^\infty d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} \int_0^\infty d\xi \xi^{n+k-1} \left(\left(\frac{d}{d\xi}\right)^k \delta(s_1 - (\xi + \eta m_Q^2))\right) \delta(s_2 - (\xi + \eta m_Q^2)). \quad (\text{A22})$$

By performing the integral over  $\xi$ , we acquire the following expression for the double spectral density:

$$\rho(s_1, s_2) = \Sigma a_k \frac{(-1)^k}{\Gamma(n+k)\Gamma(\alpha)} \int_1^\infty d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} (s_1 - \eta m_Q^2)^{n+k-1} \left( \left( \frac{d}{ds_1} \right)^k \delta(s_2 - s_1) \right) \theta(s_1 - \eta m_Q^2), \quad (\text{A23})$$

which can be written as

$$\rho(s_1, s_2) = \Sigma a_k \frac{(-1)^k}{\Gamma(n+k)\Gamma(\alpha)} \int_1^{s_1/m_Q^2} d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} (s_1 - \eta m_Q^2)^{n+k-1} \left( \left( \frac{d}{ds_1} \right)^k \delta(s_2 - s_1) \right). \quad (\text{A24})$$

With the use of this double spectral density, the continuum-subtracted correlation function in the Borel scheme corresponding to the generic term under consideration is written as

$$\Pi^{\text{sub}} = \int_{m_Q^2}^{s_0} ds_1 \int_{m_Q^2}^{s_0} ds_2 \rho(s_1, s_2) e^{-s_1/M_1^2} e^{-s_2/M_2^2}. \quad (\text{A25})$$

Now we define the new variables  $s_1 = 2sv$  and  $s_2 = 2s(1-v)$ . As a result, we obtain

$$\Pi^{\text{sub}} = \int_{m_Q^2}^{s_0} ds \int dv \rho(s_1, s_2) (4s) e^{-2sv/M_1^2} e^{-2s(1-v)/M_2^2}. \quad (\text{A26})$$

Inserting the above expression for the spectral density, one can immediately get

$$\begin{aligned} \Pi^{\text{sub}} &= \Sigma a_k \frac{(-1)^k}{\Gamma(n+k)\Gamma(\alpha)} \int_{m_Q^2}^{s_0} ds \int dv \frac{1}{2^k s^k} \left( \left( \frac{d}{dv} \right)^k \delta(v - 1/2) \right) \\ &\times \int_1^{2sv/m_Q^2} d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} (2sv - \eta m_Q^2)^{n+k-1} e^{-2sv/M_1^2} e^{-2s(1-v)/M_2^2}. \end{aligned} \quad (\text{A27})$$

Now we perform the integration over  $v$ , which leads to the final form

$$\begin{aligned} \Pi^{\text{sub}} &= \Sigma a_k \frac{(-1)^k (-1)^k}{\Gamma(n+k)\Gamma(\alpha)} \int_{m_Q^2}^{s_0} ds \frac{1}{2^k s^k} \\ &\times \left[ \left( \frac{d}{dv} \right)^k \int_1^{2sv/m_Q^2} d\eta \eta^{\beta-\alpha-1} (\eta-1)^{\alpha-1} (2sv - \eta m_Q^2)^{n+k-1} e^{-2sv/M_1^2} e^{-2s(1-v)/M_2^2} \right]_{v=1/2}. \end{aligned} \quad (\text{A28})$$

Now we extend these calculations to all of the terms entering the expressions for the coupling constants under consideration. As the calculations are very lengthy, as an example we only present our final result for the  $\frac{3}{2}$  case and the  $\tilde{\Pi}_1^{*\text{OPE}}$  function defining the  $\Sigma_b(6097)^+ \rightarrow \Lambda_b^0 \pi^+$  transition. For this function, we get

$$\tilde{\Pi}_1^{*\text{OPE}} = \int_{m_b^2}^{s_0} e^{-\frac{s}{M^2} - \frac{m_b^2}{4M^2}} \rho_1(s) ds + e^{-\frac{m_b^2}{M^2} - \frac{m_b^2}{4M^2}} \Gamma_1, \quad (\text{A29})$$

where the expressions for  $\rho_1(s)$  and  $\Gamma_1$  are given as

$$\begin{aligned}
\rho_1(s) = & \frac{1}{96\sqrt{2}m_b^2\pi^2} [-\psi_{31}m_b^4\mu_\pi\zeta_4 + \psi_{31}m_b^4\mu_\pi\beta\zeta_4 + \psi_{20}m_b^4\mu_\pi(\beta-1)(\zeta_4-2\zeta_5) \\
& + 2\psi_{31}m_b^4\mu_\pi\zeta_5 - 2\psi_{31}m_b^4\mu_\pi\beta\zeta_5 + 12f_\pi\psi_{21}m_\pi^2m_b^3\zeta_7 + 12f_\pi\psi_{21}m_\pi^2m_b^3\beta\zeta_7 \\
& + 12f_\pi\psi_{21}m_\pi^2m_b^2m_u\beta\zeta_7 - 6f_\pi\psi_{21}m_\pi^2m_b^3\zeta_1 - 6f_\pi\psi_{21}m_\pi^2m_b^2m_u\zeta_1 - 6f_\pi\psi_{21}m_\pi^2m_b^3\beta\zeta_1 \\
& + 3f_\pi\psi_{11}m_\pi^2m_b^2m_u\beta\zeta_1 - 15f_\pi\psi_{12}m_\pi^2m_b^2m_u\beta\zeta_1 - 12f_\pi\psi_{12}m_\pi^2m_us\beta\zeta_1 - 12f_\pi\psi_{21}m_\pi^2m_b^3\zeta_2 \\
& - 2f_\pi\psi_{11}m_\pi^2m_b^2m_u\zeta_2 + 10f_\pi\psi_{12}m_\pi^2m_b^2m_u\zeta_2 - 8f_\pi\psi_{21}m_\pi^2m_b^2m_u\zeta_2 + 8f_\pi\psi_{12}m_\pi^2m_us\zeta_2 \\
& - 12f_\pi\psi_{21}m_\pi^2m_b^3\beta\zeta_2 - 4f_\pi\psi_{11}m_\pi^2m_b^2m_u\beta\zeta_2 + 20f_\pi\psi_{12}m_\pi^2m_b^2m_u\beta\zeta_2 \\
& - 4f_\pi\psi_{21}m_\pi^2m_b^2m_u\beta\zeta_2 + 16f_\pi\psi_{12}m_\pi^2m_us\beta\zeta_2 + 8f_\pi\psi_{21}m_\pi^2m_b^2(2m_b+m_u)(2+\beta)\zeta_6 \\
& + 2\psi_{10}m_b^2(2m_b^2\mu_\pi(\zeta_4+2\beta\zeta_4-(2+\beta)\zeta_5) + f_\pi m_\pi^2 m_u(6\beta\zeta_7-3\zeta_1-4\zeta_2-2\beta\zeta_2+4(2+\beta)\zeta_6) \\
& + f_\pi m_\pi^2 m_b(6(\beta+\beta)\zeta_7-3(1+\beta)(\zeta_1+2\zeta_2)+8(2+\beta)\zeta_6))] \\
& \times \left( 2(-6\beta\zeta_7+18\gamma_E\beta\zeta_7+3\zeta_1-9\gamma_E\zeta_1-3\gamma_E\beta\zeta_1+4\zeta_2-10\gamma_E\zeta_2+2\beta\zeta_2-2\gamma_E\beta\zeta_2 \right. \\
& + 4(3\gamma_E-1)(2+\beta)\zeta_6) - (18\beta\zeta_7-9\zeta_1-3\beta\zeta_1-10\zeta_2-2\beta\zeta_2 \\
& + 12(2+\beta)\zeta_6) \left( \ln\left(\frac{\Lambda^2}{m_b^2}\right) + 2\ln\left(\frac{M^2}{\Lambda^2}\right) \right) + \frac{1}{32\sqrt{2}\pi^2} f_\pi m_\pi^2 [-2\psi_{10}m_b(1+\beta) \\
& + (\psi_{20}+\psi_{31})(m_b+m_b\beta-m_u\beta) + 2m_b(1+\beta)] \varphi_\pi(u_0) \\
& + \frac{1}{48\sqrt{2}\pi^2} (-1+\tilde{\mu}_\pi^2)m_b\mu_\pi[2(\psi_{10}+\psi_{21})m_u(\beta-1) + (\psi_{20}+\psi_{31})m_b\beta] \varphi_\sigma(u_0) \\
& + \frac{\langle\bar{u}u\rangle}{6\sqrt{2}} f_\pi\beta\varphi_\pi(u_0) + \frac{\langle g^2G^2\rangle}{288\sqrt{2}m_b^4\pi^2} f_\pi m_\pi^2 m_u (-18\beta\zeta_7+9\zeta_1+3\beta\zeta_1+10\zeta_2+2\beta\zeta_2 \\
& - 12(2+\beta)\zeta_6) \left[ 2-\psi_{01}-\psi_{02}-9\psi_{10}+6\gamma_E\psi_{10}+3\psi_{21}+\psi_{22}-6\psi_{10}\ln\left(\frac{M^2}{\Lambda^2}\right) \right. \\
& + 2(1-\psi_{03}+3\psi_{21}+2\psi_{22}+\psi_{23})\ln\left(\frac{s-m_b^2}{\Lambda^2}\right) \left. \right] + \frac{\langle g^2G^2\rangle}{64\sqrt{2}m_b^6\pi^2} 5f_\pi m_\pi^2 m_u\beta\mathbb{A}(u_0) \\
& \times \left[ -m_b^2 \left( 6(2\gamma_E-3)\psi_{10}+6\psi_{21}+3\psi_{22}+\psi_{23}-4\psi_{-10}+\psi_{-13}+3\psi_{-1-1}-12\psi_{10}\ln\left(\frac{M^2}{\Lambda^2}\right) \right) \right. \\
& + 12(m_b^2-s)\ln\left(\frac{s-m_b^2}{\Lambda^2}\right) \left. \right] + \frac{\langle g^2G^2\rangle}{576\sqrt{2}m_b^4\pi^2} f_\pi \left[ (1-3(\psi_{10}+\psi_{21}))m_b^3 + (m_b^2(-3(\psi_{10}+\psi_{21})m_b \right. \\
& - 3(-2\psi_{01}-4\psi_{02}+3\psi_{03}-37\psi_{10}+36\gamma_E\psi_{10}+\psi_{12}+3\psi_{13}+6\psi_{21}+2\psi_{22}))m_u \\
& + (m_b+9(-3+4\gamma_E)m_u)) + 15m_us)\beta + 12m_u\beta \left( -3(1-3\psi_{10})m_b^2\ln\left(\frac{M^2}{\Lambda^2}\right) \right. \\
& + ((8+\psi_{03}-3\psi_{21}-2\psi_{22}-\psi_{23})m_b^2-6s)\ln\left(\frac{s-m_b^2}{\Lambda^2}\right) \left. \right) \left. \right] \varphi_\pi(u_0) \\
& - \frac{\langle g^2G^2\rangle}{48\sqrt{2}m_b^3\pi^2} (\tilde{\mu}_\pi^2-1)m_u\mu_\pi(\beta-1) \left[ 2-\psi_{01}-\psi_{02}-9\psi_{10}+6\gamma_E\psi_{10}+3\psi_{21}+\psi_{22}-6\psi_{10}\ln\left(\frac{M^2}{\Lambda^2}\right) \right. \\
& + 2(1-\psi_{03}+3\psi_{21}+2\psi_{22}+\psi_{23})\ln\left(\frac{s-m_b^2}{\Lambda^2}\right) \left. \right] \varphi_\sigma(u_0)
\end{aligned} \tag{A30}$$

and

$$\begin{aligned}
\Gamma_1 = & -\frac{\langle \bar{u}u \rangle}{432\sqrt{2}M^6} [-12M^4(m_u M^2 \mu_\pi (\zeta_4 + 2\beta\zeta_4 - (2+\beta)\zeta_5) + f_\pi m_\pi^2 m_b m_u (\beta-1)\zeta_2 + f_\pi m_\pi^2 M^2 (-6\beta\zeta_7 + 3\zeta_1 + 4\zeta_2 \\
& + 2\beta\zeta_2 - 4(2+\beta)\zeta_6)) + m_o^2 m_b (2f_\pi m_\pi^2 m_b^2 m_u (\beta-1)\zeta_2 - 2f_\pi m_\pi^2 m_u M^2 (\beta-1)\zeta_2 + m_b M^2 (2m_u \mu_\pi (\zeta_4 + 2\beta\zeta_4 \\
& - (2+\beta)\zeta_5) + 3f_\pi m_\pi^2 (-6\beta\zeta_7 + 3\zeta_1 + 4\zeta_2 + 2\beta\zeta_2 - 4(2+\beta)\zeta_6))] + \frac{\langle \bar{u}u \rangle}{1728\sqrt{2}M^8} f_\pi m_\pi^2 [36M^4 (-2m_b^2 M^2 \beta - 2M^4 \beta \\
& + m_b^3 m_u (1+\beta)) + m_o^2 m_b (2m_u M^4 (\beta-1) + 18m_b^3 M^2 \beta - 6m_b^4 m_u (1+\beta) + m_b M^4 (5+4\beta) \\
& + 2m_b^2 m_u M^2 (7+5\beta))] \mathbb{A}(u_0) - \frac{\langle \bar{u}u \rangle}{432\sqrt{2}M^4} f_\pi [36m_b m_u M^4 (1+\beta) + m_o^2 (-2m_b m_u M^2 (\beta-1) + 18m_b^2 M^2 \beta - 6m_b^3 m_u (1+\beta) \\
& + M^4 (5+22\beta))] \varphi_\pi(u_0) - \frac{\langle \bar{u}u \rangle}{432\sqrt{2}M^6} (\tilde{\mu}_\pi - 1)(1 + \tilde{\mu}_\pi) \mu_\pi [-12M^4 (-2m_b M^2 (\beta-1) + m_b^2 m_u \beta + m_u M^2 \beta) \\
& + m_o^2 m_b^2 (-6m_b M^2 (\beta-1) + 2m_b^2 m_u \beta + m_u M^2 (2\beta-1))] \varphi_\sigma(u_0) + \frac{\langle g^2 G^2 \rangle}{3456\sqrt{2}\pi^2} \left[ \frac{1}{M^2} f_\pi m_\pi^2 (m_b (-6(1+\beta)\zeta_7 \right. \\
& + 3(1+\beta)\zeta_1 + 4(2+\beta)(\zeta_2 - 2\zeta_6)) + 2m_u (-18\beta\zeta_7 + 9\zeta_1 + 3\beta\zeta_1 + 10\zeta_2 + 2\beta\zeta_2 - 12(2+\beta)\zeta_6) \\
& + \frac{1}{m_b^2} (-m_b^2 \mu_\pi (\zeta_4 + 5\beta\zeta_4 - 2(\zeta_5 + 2\beta\zeta_5)) + 2f_\pi m_\pi^2 m_b (6(1+\beta)\zeta_7 - 3(1+\beta)\zeta_1 - 4(2+\beta)(\zeta_2 - 2\zeta_6)) \\
& + f_\pi m_\pi^2 m_u (-18\beta\zeta_7 + 9\zeta_1 + 3\beta\zeta_1 + 10\zeta_2 + 2\beta\zeta_2 - 12(2+\beta)\zeta_6) + \frac{1}{M^4} f_\pi m_\pi^2 m_b^2 m_u \left( 2(-6\beta\zeta_7 + 18\gamma_E \beta\zeta_7 \right. \\
& + 3\zeta_1 - 9\gamma_E \zeta_1 - 3\gamma_E \beta\zeta_1 + 4\zeta_2 - 10\gamma_E \zeta_2 + 2\beta\zeta_2 - 2\gamma_E \beta\zeta_2 + 4(3\gamma_E - 1)(2+\beta)\zeta_6) - (18\beta\zeta_7 - 9\zeta_1 - 3\beta\zeta_1 \\
& - 10\zeta_2 - 2\beta\zeta_2 + 12(2+\beta)\zeta_6) \left. \left[ \ln\left(\frac{\Lambda^2}{m_b^2}\right) + 2\ln\left(\frac{M^2}{\Lambda^2}\right) \right] \right] + \frac{\langle g^2 G^2 \rangle}{2304\sqrt{2}m_b^2 M^6 \pi^2} f_\pi m_\pi^2 \mathbb{A}(u_0) \left[ 2(3\gamma_E - 1)m_b^6 m_u \beta \right. \\
& - 10m_b^4 m_u M^2 \beta - 9m_b^2 m_u M^4 \beta - 6m_u M^6 \beta - m_b^3 M^4 (1+\beta) + 2m_b M^6 (1+\beta) - 3m_b^6 m_u \beta \left. \left[ \ln\left(\frac{\Lambda^2}{mqb^2}\right) + 2\ln\left(\frac{M^2}{\Lambda^2}\right) \right] \right] \\
& + \frac{\langle g^2 G^2 \rangle}{576\sqrt{2}M^2 \pi^2} f_\pi m_u \beta \left[ (2 - 6\gamma_E)m_b^2 - 6(\gamma_E - 1)M^2 + 3(m_b^2 + 2M^2) \ln\left(\frac{\Lambda^2}{m_b^2}\right) + 3(2m_b^2 + 3M^2) \ln\left(\frac{M^2}{\Lambda^2}\right) \right] \varphi_\pi(u_0) \\
& + \frac{\langle g^2 G^2 \rangle}{1728\sqrt{2}m_b M^4 \pi^2} (\tilde{\mu}_\pi^2 - 1) \mu_\pi \left[ 2(3\gamma_E - 1)m_b^4 m_u (\beta-1) - 7m_b^2 m_u M^2 (\beta-1) - m_u M^4 (\beta-1) - m_b M^4 \beta \right. \\
& - 3m_b^4 m_u (\beta-1) \left. \left[ \ln\left(\frac{\Lambda^2}{m_b^2}\right) + 2\ln\left(\frac{M^2}{\Lambda^2}\right) \right] \right] \varphi_\sigma(u_0) + \frac{\langle g^2 G^2 \rangle \langle \bar{u}u \rangle}{20736\sqrt{2}M^{14}} f_\pi m_\pi^2 m_b [-6M^4 (-2m_b^3 M^2 \beta + 4m_b M^4 \beta \\
& + m_b^4 m_u (1+\beta) - 6m_b^2 m_u M^2 (1+\beta) + 6m_u M^4 (1+\beta)) + m_o^2 (-3m_b^3 M^2 \beta + 18m_b^3 M^4 \beta - 18m_b M^6 \beta \\
& + m_b^6 m_u (1+\beta) - 11m_b^4 m_u M^2 (1+\beta) + 30m_b^2 m_u M^4 (1+\beta) - 18m_u M^6 (1+\beta))] \mathbb{A}(u_0) - \frac{\langle g^2 G^2 \rangle \langle \bar{u}u \rangle}{5184\sqrt{2}M^{10}} f_\pi m_b [6M^4 (2m_b M^2 \beta \\
& - m_b^2 m_u (1+\beta) + 3m_u M^2 (1+\beta)) + m_o^2 (-3m_b^3 M^2 \beta + 6m_b M^4 \beta + m_b^4 m_u (1+\beta) - 6m_b^2 m_u M^2 (1+\beta) \\
& + 6m_u M^4 (1+\beta))] \varphi_\pi(u_0) + \frac{\langle g^2 G^2 \rangle \langle \bar{u}u \rangle}{15552\sqrt{2}M^{12}} (\tilde{\mu}_\pi^2 - 1) m_b \mu_\pi [m_o^2 (m_b^4 - 6m_b^2 M^2 + 6M^4) (-3M^2 (\beta-1) + m_b m_u \beta) \\
& - 6M^4 (2(m_b^2 - 3M^2)M^2 + (m_b^3 m_u - 2m_b (m_b + m_u)M^2 + 6M^4)\beta)] \varphi_\sigma(u_0). \tag{A31}
\end{aligned}$$

In the above functions,  $\zeta_j$  and  $\psi_{nm}$  are defined as

$$\begin{aligned}
\zeta_j &= \int \mathcal{D}\alpha_i \int_0^1 dv f_j(\alpha_i) \delta(k(\alpha_q + v\alpha_g) - u_0), \\
\psi_{nm} &= \frac{(s - m_Q^2)^n}{s^m (m_Q^2)^{n-m}}, \tag{A32}
\end{aligned}$$

with the distribution amplitudes that are given as  $f_1(\alpha_i) = \mathcal{V}_\parallel(\alpha_i)$ ,  $f_2(\alpha_i) = \mathcal{V}_\perp(\alpha_i)$ ,  $f_3(\alpha_i) = \mathcal{A}_\parallel(\alpha_i)$ ,  $f_4(\alpha_i) = \mathcal{T}(\alpha_i)$ ,  $f_5(\alpha_i) = v\mathcal{T}(\alpha_i)$ ,  $f_6(\alpha_i) = v\mathcal{V}_\perp(\alpha_i)$ ,  $f_7(\alpha_i) = v\mathcal{A}_\parallel(\alpha_i)$ , whose explicit forms can be found in Refs. [18–20]. As we previously mentioned,  $u_0$  has the form  $u_0 = \frac{M_1^2}{M_1^2 + M_2^2}$ . Considering the similar masses of the initial and final baryons and taking  $M_1^2 = M_2^2$ , it becomes  $u_0 = \frac{1}{2}$ . In the above results,  $\mu_\pi = f_\pi \frac{m_\pi^2}{m_u + m_d}$ ,  $\tilde{\mu}_\pi = \frac{m_u + m_d}{m_\pi}$ , and  $\mathcal{D}\alpha = d\alpha_{\bar{q}} d\alpha_q d\alpha_g \delta(1 - \alpha_{\bar{q}} - \alpha_q - \alpha_g)$  are used. The functions  $\varphi_\pi(u)$ ,  $\mathbb{A}(u)$ ,  $\mathbb{B}(u)$ ,  $\varphi_P(u)$ ,  $\varphi_\sigma(u)$ ,  $\mathcal{T}(\alpha_i)$ ,  $\mathcal{A}_\perp(\alpha_i)$ ,  $\mathcal{A}_\parallel(\alpha_i)$ ,  $\mathcal{V}_\perp(\alpha_i)$ , and  $\mathcal{V}_\parallel(\alpha_i)$  are functions of definite twists, which can also be found in Refs. [18–20]. They are given as

$$\begin{aligned}
\varphi_\pi(u) &= 6u\bar{u} \left( 1 + a_1^\pi C_1(2u-1) + a_2^\pi C_2^{\frac{3}{2}}(2u-1) \right), \\
\mathcal{T}(\alpha_i) &= 360\eta_3 \alpha_{\bar{q}} \alpha_q \alpha_g^2 \left( 1 + w_3 \frac{1}{2}(7\alpha_g - 3) \right), \\
\varphi_P(u) &= 1 + \left( 30\eta_3 - \frac{5}{2}\mu_\pi^2 \right) C_2^{\frac{1}{2}}(2u-1) \\
&\quad + \left( -3\eta_3 w_3 - \frac{27}{20}\mu_\pi^2 - \frac{81}{10}\mu_\pi^2 a_2^\pi \right) C_4^{\frac{1}{2}}(2u-1), \\
\varphi_\sigma(u) &= 6u\bar{u} \left[ 1 + \left( 5\eta_3 - \frac{1}{2}\eta_3 w_3 - \frac{7}{20}\mu_\pi^2 - \frac{3}{5}\mu_\pi^2 a_2^\pi \right) C_2^{\frac{3}{2}}(2u-1) \right], \\
\mathcal{V}_\parallel(\alpha_i) &= 120\alpha_q \alpha_{\bar{q}} \alpha_g (v_{00} + v_{10}(3\alpha_g - 1)), \\
\mathcal{A}_\parallel(\alpha_i) &= 120\alpha_q \alpha_{\bar{q}} \alpha_g (0 + a_{10}(\alpha_q - \alpha_{\bar{q}})), \\
\mathcal{V}_\perp(\alpha_i) &= -30\alpha_g^2 \left[ h_{00}(1 - \alpha_g) + h_{01}(\alpha_g(1 - \alpha_g) - 6\alpha_q \alpha_{\bar{q}}) + h_{10} \left( \alpha_g(1 - \alpha_g) - \frac{3}{2}(\alpha_{\bar{q}}^2 + \alpha_q^2) \right) \right], \\
\mathcal{A}_\perp(\alpha_i) &= 30\alpha_g^2 (\alpha_{\bar{q}} - \alpha_q) \left[ h_{00} + h_{01}\alpha_g + \frac{1}{2}h_{10}(5\alpha_g - 3) \right], \\
\mathbb{B}(u) &= g_\pi(u) - \phi_\pi(u), \\
g_\pi(u) &= g_0 C_0^{\frac{1}{2}}(2u-1) + g_2 C_2^{\frac{1}{2}}(2u-1) + g_4 C_4^{\frac{1}{2}}(2u-1), \\
\mathbb{A}(u) &= 6u\bar{u} \left[ \frac{16}{15} + \frac{24}{35}a_2^\pi + 20\eta_3 + \frac{20}{9}\eta_4 + \left( -\frac{1}{15} + \frac{1}{16} - \frac{7}{27}\eta_3 w_3 - \frac{10}{27}\eta_4 \right) C_2^{\frac{3}{2}}(2u-1) \right. \\
&\quad \left. + \left( -\frac{11}{210}a_2^\pi - \frac{4}{135}\eta_3 w_3 \right) C_4^{\frac{3}{2}}(2u-1) \right] \\
&\quad + \left( -\frac{18}{5}a_2^\pi + 21\eta_4 w_4 \right) [2u^3(10 - 15u + 6u^2) \ln u \\
&\quad + 2\bar{u}^3(10 - 15\bar{u} + 6\bar{u}^2) \ln \bar{u} + u\bar{u}(2 + 13u\bar{u})], \tag{A33}
\end{aligned}$$

where  $C_n^k(x)$  are the Gegenbauer polynomials,

$$\begin{aligned}
h_{00} = v_{00} &= -\frac{1}{3}\eta_4, & h_{01} &= \frac{7}{4}\eta_4 w_4 - \frac{3}{20}a_2^\pi, & h_{10} &= \frac{7}{4}\eta_4 w_4 + \frac{3}{20}a_2^\pi, & a_{10} &= \frac{21}{8}\eta_4 w_4 - \frac{9}{20}a_2^\pi, \\
v_{10} &= \frac{21}{8}\eta_4 w_4, & g_0 &= 1, & g_2 &= 1 + \frac{18}{7}a_2^\pi + 60\eta_3 + \frac{20}{3}\eta_4, & g_4 &= -\frac{9}{28}a_2^\pi - 6\eta_3 w_3. \tag{A34}
\end{aligned}$$

The constants presented in Eqs. (A33) and (A34) are calculated using QCD sum rules at the renormalization scale  $\mu = 1 \text{ GeV}^2$  [18–20,25–29]. These constants are given as  $a_1^\pi = 0$ ,  $a_2^\pi = 0.44$ ,  $\eta_3 = 0.015$ ,  $\eta_4 = 10$ ,  $w_3 = -3$ , and  $w_4 = 0.2$ .



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