



# Integrable double deformation of the principal chiral model

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## Abstract

We define a two-parameter family of integrable deformations of the principal chiral model on an arbitrary compact group. The Yang–Baxter  $\sigma$ -model and the principal chiral model with a Wess–Zumino term both correspond to limits in which one of the two parameters vanishes.

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## 1. Introduction

It was shown by S. Rajeev in [1] that there exists a one-parameter deformation of the Poisson brackets satisfied by the current of the principal chiral model. It was also observed that this deformed Poisson structure coincides with that of two Poisson commuting classical Kac–Moody currents [1]. It turns out that the Kac–Moody currents are either both real or complex conjugate of one another, depending on the value of the deformation parameter [2]. We will therefore refer to these two branches in the deformation parameter as real and complex, respectively. In each branch, the integrable field theory which provides a realisation of the deformed Poisson algebra

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is known. As was shown in [3], for the complex branch this is the Yang–Baxter  $\sigma$ -model defined by C. Klimčík in [4,5]. The Yang–Baxter  $\sigma$ -model on  $SU(2)$  is simply the squashed 3-sphere  $\sigma$ -model studied in [6,7]. For the real branch, the model is the one introduced by K. Sfetsos in [8] (see also [2,9,10]). It generalises the model studied in [11] to higher rank.

In the case of  $SU(2)$ , it was found in [11] that the deformed Poisson brackets can even be extended to a two-parameter deformation. Furthermore, it is clear that the Poisson brackets constructed in [11] extend immediately to any Lie algebra. In this note we exhibit the action of the integrable field theory which realises the double deformation in the complex branch.

## 2. Action and Lax pair

### 2.1. Ansatz for the action

Let  $\mathfrak{g}$  be a compact Lie algebra with Lie group  $G$ . To construct a two-parameter deformation of the principal chiral model on  $G$  we will start with a fairly general ansatz for the action of a  $G$ -valued field  $g$ , which we take to be:

$$S[g] = -\frac{1}{2}K \int d^2x \operatorname{tr} \left( g^{-1} \partial_- g \frac{1 + \eta^2}{1 - \eta^2 R^2} g^{-1} \partial_+ g + g^{-1} \partial_- g \frac{(1 + \eta^2)AR}{1 - \eta^2 R^2} g^{-1} \partial_+ g \right) - \frac{1}{2}kK \int d^3x \operatorname{tr} (g^{-1} \partial_\xi g [g^{-1} \partial_- g, g^{-1} \partial_+ g]). \tag{2.1}$$

Here  $\partial_\pm = \partial_\tau \pm \partial_\sigma$  denote the usual light-cone derivatives on the worldsheet. The last term in (2.1) is the standard Wess–Zumino term integrated over a 3-dimensional space parameterised by  $(\tau, \sigma, \xi)$  and whose boundary is the worldsheet. We denote this term by  $S_{WZ}[g]$ .

For the moment  $\eta, A$  and  $k$  are three independent real parameters. The real parameter  $K$  is an overall normalisation which will not play much role in our analysis. Note that when  $k = 0$  and  $A = \pm\eta$  one recovers the Yang–Baxter  $\sigma$ -model of [4,5]. Specifically, the first two terms in (2.1) reduce in this limit to the symmetric and anti-symmetric part of the operator  $(1 \mp \eta R)^{-1}$  in the Yang–Baxter  $\sigma$ -model action respectively. Furthermore, when  $\eta = A = k = 0$  the above action reduces to that of the principal chiral model.

The linear operator  $R$  is a skew-symmetric non-split solution of the modified classical Yang–Baxter equation (see [3] for details) on  $\mathfrak{g}$ . That is to say, for any  $x, y \in \mathfrak{g}$  it satisfies

$$[Rx, Ry] = R([Rx, y] + [x, Ry]) + [x, y]. \tag{2.2}$$

In what follows we shall work with the standard solution to this equation which is constructed as follows. Given a Cartan–Weyl basis  $(H^i, E^\alpha)$  of the complexified Lie algebra  $\mathfrak{g}^{\mathbb{C}}$ , a basis of the compact real form  $\mathfrak{g}$  is given by

$$T^i = iH^i, \quad B^\alpha = \frac{i}{\sqrt{2}}(E^{+\alpha} + E^{-\alpha}), \quad C^\alpha = \frac{1}{\sqrt{2}}(E^{+\alpha} - E^{-\alpha}).$$

The operator  $R$  is then defined by [5]

$$R(T^i) = 0, \quad R(B^\alpha) = C^\alpha, \quad R(C^\alpha) = -B^\alpha. \tag{2.3}$$

2.2. Flat and conserved current

The equations of motion computed from the action (2.1) take the form of the conservation equation

$$\partial_- \mathcal{K}_+ + \partial_+ \mathcal{K}_- = 0$$

where we have defined

$$\mathcal{K}_\pm = g \left( \left( \frac{1 + \eta^2}{1 - \eta^2 R^2} (1 \pm AR) \mp k \right) g^{-1} \partial_\pm g \right) g^{-1}. \tag{2.4}$$

We will now determine the conditions under which this conserved current  $\mathcal{K}_\pm$  is also on-shell flat. This will immediately imply the existence of a Lax pair for the resulting model. However, for simplicity, we shall make use of the fact that by construction the  $R$ -matrix (2.3) satisfies

$$R^3 = -R. \tag{2.5}$$

Using the property (2.5) one can show that

$$\frac{1 + \eta^2}{1 - \eta^2 R^2} = 1 + \eta^2 + \eta^2 R^2, \quad \frac{(1 + \eta^2)R}{1 - \eta^2 R^2} = R. \tag{2.6}$$

The action (2.1) may therefore be written as

$$S[g] = -\frac{1}{2} K \int d^2x \operatorname{tr}(g^{-1} \partial_- g (1 + \eta^2 + AR + \eta^2 R^2) g^{-1} \partial_+ g) + S_{\text{WZ}}[g]. \tag{2.7}$$

Substituting (2.6) into the above expression (2.4) for  $\mathcal{K}_\pm$  we obtain

$$g^{-1} \mathcal{K}_\pm g = (1 + \eta^2 \mp k \pm AR + \eta^2 R^2) g^{-1} \partial_\pm g. \tag{2.8}$$

The inverse of the operator on the right hand side of (2.8) can be constructed explicitly by equating the coefficients in front of each power of  $R$ , using (2.5), on both sides of the following equation

$$(a_\pm + b_\pm R + c_\pm R^2)(1 + \eta^2 \mp k \pm AR + \eta^2 R^2) = 1.$$

We find explicitly that

$$\begin{aligned} a_\pm &= \frac{1}{1 + \eta^2 \mp k}, & b_\pm &= \mp \frac{A}{A^2 + (1 \mp k)^2}, \\ c_\pm &= \frac{1}{1 + \eta^2 \mp k} - \frac{1 \mp k}{A^2 + (1 \mp k)^2}. \end{aligned} \tag{2.9}$$

This inverse operator enables us to rewrite (2.8) as

$$\partial_\pm g g^{-1} = (a_\pm + b_\pm R_g + c_\pm R_g^2) \mathcal{K}_\pm, \tag{2.10}$$

where the action of the operator  $R_g$  on any  $x \in \mathfrak{g}$  is defined as

$$R_g x = g R(g^{-1} x g) g^{-1}. \tag{2.11}$$

Next we note that the modified classical Yang–Baxter equation (2.2) for  $R$  together with the property (2.5) implies the following equation for  $R^2$ ,

$$[R^2 x, R^2 y] = R^2([R^2 x, y] + [x, R^2 y]) + (1 + 2R^2)[x, y]. \tag{2.12}$$

Substituting the expression (2.10) into the curvature

$$\partial_-(\partial_+gg^{-1}) - \partial_+(\partial_-gg^{-1}) + [\partial_+gg^{-1}, \partial_-gg^{-1}]$$

and making use of the modified classical Yang–Baxter equation (2.2) for the operator  $R$  as well as its consequence (2.12), one can show that

$$\begin{aligned} &\partial_-(\partial_+gg^{-1}) - \partial_+(\partial_-gg^{-1}) + [\partial_+gg^{-1}, \partial_-gg^{-1}] \\ &= \left(\frac{a_+ - a_-}{2} + \frac{b_+ - b_-}{2}R_g + \frac{c_+ - c_-}{2}R_g^2\right)(\partial_-K_+ + \partial_+K_-) \\ &\quad + \left(\frac{a_+ + a_-}{2} + \frac{b_+ + b_-}{2}R_g + \frac{c_+ + c_-}{2}R_g^2\right)(\partial_-K_+ - \partial_+K_-) \\ &\quad + (a_+a_- - b_+b_- - c_+c_- + ((a_+ - c_+)b_- + (a_- - c_-)b_+)R_g \\ &\quad + ((a_+ - c_+)c_- + (a_- - c_-)c_+)R_g^2)[K_+, K_-] \\ &\quad + b_+c_-(1 + R_g^2)[R_gK_-, K_+] + b_-c_+(1 + R_g^2)[K_-, R_gK_+]. \end{aligned} \tag{2.13}$$

Now we have the following relations between the parameters (2.9),

$$\frac{b_+ + b_-}{2} = (a_+ - c_+)b_- + (a_- - c_-)b_+. \tag{2.14}$$

Moreover, we impose the relation

$$A = \eta\sqrt{1 - \frac{k^2}{1 + \eta^2}}, \tag{2.15}$$

between the three parameters  $\eta$ ,  $A$  and  $k$ . Note that the exact same relation (2.15) was previously obtained in [12,13] for the ‘squashed WZNW-model’, which is a deformation of the squashed 3-sphere  $\sigma$ -model by addition of a WZ term. The model defined by the action (2.1), with  $A$  set to the value (2.15), can therefore be seen as a generalisation of the latter from the  $\mathfrak{su}(2)$  case to an arbitrary Lie algebra  $\mathfrak{g}$ . Using (2.15), we find the following further relations among these parameters

$$b_+c_- = b_-c_+, \tag{2.16a}$$

$$\frac{a_+ + a_-}{2} = a_+a_- - b_+b_- - c_+c_-, \tag{2.16b}$$

$$\frac{c_+ + c_-}{2} = (a_+ - c_+)c_- + (a_- - c_-)c_+. \tag{2.16c}$$

Substituting all these relations into (2.13) we arrive at the following

$$\begin{aligned} &\partial_-(\partial_+gg^{-1}) - \partial_+(\partial_-gg^{-1}) + [\partial_+gg^{-1}, \partial_-gg^{-1}] \\ &= \left(\frac{a_+ - a_-}{2} + \frac{b_+ - b_-}{2}R_g + \frac{c_+ - c_-}{2}R_g^2\right)(\partial_-K_+ + \partial_+K_-) \\ &\quad + \left(\frac{a_+ + a_-}{2} + \frac{b_+ + b_-}{2}R_g + \frac{c_+ + c_-}{2}R_g^2\right)(\partial_-K_+ - \partial_+K_- + [K_+, K_-]) \\ &\quad + b_+c_-(1 + R_g^2)([R_gK_-, K_+] + [K_-, R_gK_+]). \end{aligned} \tag{2.17}$$

The last line can be shown to vanish using the following identity

$$(1 + R^2)([Rx, y] + [x, Ry]) = 0, \tag{2.18}$$

valid for any  $x, y \in \mathfrak{g}$ . To show the latter, note that the left hand side can be rewritten as

$$(1 + R^2)([Rx, y] + [x, Ry]) = (R - i)[(R + i)x, (R + i)y],$$

by using the modified classical Yang–Baxter equation (2.2) for  $R$  in the form

$$(R + i)([Rx, y] + [x, Ry]) = [(R + i)x, (R + i)y].$$

However, since  $R \pm i$  are respectively the projectors onto the Borel subalgebras  $\mathfrak{b}_\pm$  of  $\mathfrak{g}^\mathbb{C}$ , it follows that  $(R - i)[(R + i)x, (R + i)y] = 0$  for any  $x, y \in \mathfrak{g}$ , and hence we obtain (2.18). We are therefore left with

$$\begin{aligned} & \partial_- (\partial_+ g g^{-1}) - \partial_+ (\partial_- g g^{-1}) + [\partial_+ g g^{-1}, \partial_- g g^{-1}] \\ &= \left( \frac{a_+ - a_-}{2} + \frac{b_+ - b_-}{2} R_g + \frac{c_+ - c_-}{2} R_g^2 \right) (\partial_- \mathcal{K}_+ + \partial_+ \mathcal{K}_-) \\ &+ \left( \frac{a_+ + a_-}{2} + \frac{b_+ + b_-}{2} R_g + \frac{c_+ + c_-}{2} R_g^2 \right) (\partial_- \mathcal{K}_+ - \partial_+ \mathcal{K}_- + [\mathcal{K}_+, \mathcal{K}_-]). \end{aligned} \tag{2.19}$$

Since the operator in parenthesis in the last line is invertible, this implies that the conserved current  $\mathcal{K}_\pm$  is also on-shell flat. The equations of motion for the action (2.1) with the parameter  $A$  fixed by the relation (2.15) can therefore be recast in the form of a zero curvature equation for the Lax pair

$$\mathcal{L}_\pm(z) = \frac{\mathcal{K}_\pm}{1 \mp z}.$$

We denote the spatial component  $\frac{1}{2}(\mathcal{L}_+(z) - \mathcal{L}_-(z))$  by

$$\mathcal{L}(z) = \frac{1}{1 - z^2} (\mathcal{K}_1 + z\mathcal{K}_0). \tag{2.20}$$

### 2.3. 3-parameter current

Before studying the 2-parameter action (2.1) with  $A$  fixed by (2.15), let us note that there is a redundancy between the set of four equations (2.14) and (2.16) for the six coefficients  $(a_\pm, b_\pm, c_\pm)$  entering in Eq. (2.10). Indeed, Eqs. (2.14) and (2.16a) together imply (2.16c). As a consequence, it is possible to construct a 3-parameter current<sup>1</sup> whose conservation equation implies its flatness. After solving the remaining independent equations, one may parameterise this current as

$$g^{-1} \mathcal{K}_\pm g = (1 + \eta^2 \mp k \pm AR + \eta^2 R^2 \pm \xi R^2) g^{-1} \partial_\pm g, \tag{2.21}$$

where  $A$  is now given in terms of  $k, \eta$  and  $\xi$  as

$$A = \eta \sqrt{1 - \frac{\frac{\xi^2}{\eta^2} + (k + \xi)^2}{1 + \eta^2}}.$$

It is not clear, however, which action could give rise to such a current.

<sup>1</sup> We thank B. Hoare for pointing out this possibility.

### 3. Double deformation of the Poisson brackets

In this section, we will derive the Hamiltonian form of the fields  $\mathcal{K}_0, \mathcal{K}_1$  and show that their Poisson brackets correspond to a double deformation of those of the principal chiral model. They correspond to the Poisson brackets found in [11] and to the ones in [13] for the  $\mathfrak{su}(2)$  case. We show that the Poisson bracket of the Lax matrix (2.20) takes the standard  $r/s$ -form [14,15] and identify the corresponding twist function [16].

#### 3.1. Canonical analysis

To perform the canonical analysis of the action (2.1) we introduce coordinates  $\varphi^i$  on the group  $G$  and write

$$g^{-1}\partial_i g = L_i^A T_A,$$

in terms of a basis  $\{T_A\}$  of  $\mathfrak{g}$ . Letting  $f_{AB}^C$  denote the structure constants with respect to this basis, namely  $[T_A, T_B] = f_{AB}^C T_C$ , we define

$$\eta_{AB} = -\text{tr}(T_A T_B), \quad f_{ABC} = \eta_{CD} f_{AB}^D.$$

We also introduce the tensor  $\lambda_{ij} = -\lambda_{ji}$  through the relation

$$-\text{tr}(g^{-1}\partial_i g [g^{-1}\partial_j g, g^{-1}\partial_k g]) = \partial_i \lambda_{jk} + \partial_j \lambda_{ki} + \partial_k \lambda_{ij}.$$

In terms of this, the WZ term in the action (2.1) can be rewritten as

$$S_{\text{WZ}}[g] = kK \int d^2x \partial_0 \varphi^i \partial_1 \varphi^j \lambda_{ij}.$$

The conjugate momenta  $\pi_i$  of  $\varphi^i$  can be computed from the action (2.1) written in the form

$$\begin{aligned} S[g] = & -\frac{1}{2}K \int d^2x \text{tr}(g^{-1}\partial_0 g (1 + \eta^2 + \eta^2 R^2) g^{-1}\partial_0 g \\ & - g^{-1}\partial_1 g (1 + \eta^2 + \eta^2 R^2) g^{-1}\partial_1 g + 2g^{-1}\partial_0 g A R g^{-1}\partial_1 g) \\ & + kK \int d^2x \partial_0 \varphi^i \partial_1 \varphi^j \lambda_{ij}. \end{aligned}$$

It is convenient to express the result in terms of the  $\mathfrak{g}$ -valued field

$$X = L_A^i \pi_i \eta^{AB} T_B,$$

where  $L_A^i$  is defined as the inverse of  $L_i^A$ , namely  $L_A^i L_i^B = \delta_A^B$ . Explicitly, we find that

$$X = K(1 + \eta^2 + \eta^2 R^2)g^{-1}\partial_0 g + K A R g^{-1}\partial_1 g + X_{\text{WZ}} \tag{3.1}$$

where

$$X_{\text{WZ}} = kK \lambda_{ij} \partial_1 \varphi^j L_A^i T^A. \tag{3.2}$$

Note that using this last expression, the WZ term in the action (2.1) can be written more succinctly as

$$S_{\text{WZ}}[g] = - \int d^2x \text{tr}(g^{-1}\partial_0 g X_{\text{WZ}}).$$

The canonical Poisson brackets between the coordinates  $\varphi^i$  and their conjugate momenta  $\pi_i$  may be conveniently expressed in terms of the fields  $g$  and  $X$  as

$$\{g_1, g_2\} = 0, \tag{3.3a}$$

$$\{X_1, g_2\} = -g_2 C_{12} \delta_{\sigma\sigma'}, \tag{3.3b}$$

$$\{X_1, X_2\} = [C_{12}, X_2] \delta_{\sigma\sigma'}, \tag{3.3c}$$

where to simplify the notation we use the convention that the argument of a function in the first (resp. second) tensor factor is  $\sigma$  (resp.  $\sigma'$ ). For instance,  $g_1 = g(\sigma) \otimes 1$  and  $g_2 = 1 \otimes g(\sigma')$ . The quadratic Casimir is  $C_{12} = \eta^{AB} T_A \otimes T_B$ . We also note the following relation

$$\{X_1, X_{WZ2}\} + \{X_{WZ1}, X_2\} = kK [C_{12}, g_2^{-1} \partial_\sigma g_2] \delta_{\sigma\sigma'} + [C_{12}, X_{WZ2}] \delta_{\sigma\sigma'}. \tag{3.3d}$$

In what follows it will be convenient to work with the  $g$ -valued field  $Y = X - X_{WZ}$ . Its Poisson brackets can be determined using (3.3) and read

$$\{Y_1, g_2\} = -g_2 C_{12} \delta_{\sigma\sigma'}, \tag{3.4a}$$

$$\{Y_1, Y_2\} = [C_{12}, Y_2] \delta_{\sigma\sigma'} - kK [C_{12}, g_2^{-1} \partial_\sigma g_2] \delta_{\sigma\sigma'}. \tag{3.4b}$$

Finally, the expression for the Hamiltonian is obtained as the Legendre transform of the Lagrangian. Explicitly we find

$$H = -\frac{K}{2(A^2 + (k - 1)^2)(A^2 + (k + 1)^2)} \int d\sigma \operatorname{tr}[(1 + k^2 + A^2)(\mathcal{K}_0^2 + \mathcal{K}_1^2) + 4k\mathcal{K}_0\mathcal{K}_1]. \tag{3.5}$$

### 3.2. Current

By using the relation (3.1) we may express  $g^{-1} \partial_0 g$  in terms of  $g^{-1} \partial_1 g$  and  $Y$ . Substituting this into the expressions (2.4) we find

$$\mathcal{K}_0 = \frac{1}{K} g Y g^{-1} - k \partial_1 g g^{-1}, \tag{3.6a}$$

$$\begin{aligned} \mathcal{K}_1 = g & \left( \frac{1}{K} \left( -\frac{k}{1 + \eta^2} + AR + \frac{k\eta^2}{1 + \eta^2} R^2 \right) Y \right. \\ & \left. + k \left( \frac{1 + \eta^2}{k} + AR + \frac{k\eta^2}{1 + \eta^2} R^2 \right) g^{-1} \partial_1 g \right) g^{-1}. \end{aligned} \tag{3.6b}$$

There are two interesting special limits of (3.6). The first one is the Yang–Baxter limit obtained by taking  $k = 0$ , which implies  $X_{WZ} = 0$ . By virtue of (2.15) we then have  $A = \eta$ . So in this limit the expressions (3.6) become

$$\mathcal{K}_0 = \frac{1}{K} g X g^{-1}, \tag{3.7a}$$

$$\mathcal{K}_1 = \frac{\eta}{K} g R X g^{-1} + (1 + \eta^2) \partial_1 g g^{-1}. \tag{3.7b}$$

This is in agreement with the expressions found in [3] provided we set  $K = 1 + \eta^2$ , which corresponds to the normalisation of the action used there. The second interesting limit corresponds to taking  $\eta = 0$  in which case  $A = 0$  and (3.6) reduce to

$$\mathcal{K}_0 = \frac{1}{K}gYg^{-1} - k\partial_1gg^{-1}, \tag{3.8a}$$

$$\mathcal{K}_1 = -\frac{k}{K}gYg^{-1} + \partial_1gg^{-1}. \tag{3.8b}$$

Finally, the  $G_L \times G_R$  invariance of the principal chiral model is broken down to  $G_L \times H_R$  by the deformation, where  $H$  is the Cartan subgroup of  $G$ . Note that  $\int d\sigma \mathcal{K}_0$  is the charge which generates the unbroken symmetry  $G_L$ . This preservation of the  $G_L$  symmetry is in contrast with the situation in the bi-Yang–Baxter  $\sigma$ -model [17], where both  $G_L$  and  $G_R$  symmetries are broken.

### 3.3. Two-parameter deformed Poisson brackets

Given the expressions (3.6) for the fields  $\mathcal{K}_0, \mathcal{K}_1$  in terms of  $g$  and  $Y$ , we may compute their Poisson brackets using (3.4). After a direct but lengthy calculation using the modified classical Yang–Baxter equation (2.2) for  $R$ , its consequence (2.12) for  $R^2$  and the expression (2.15) for  $A$  together with the identities (2.18) and

$$(1 + R^2)[x, (1 + R^2)y] = 0,$$

valid for any  $x, y \in \mathfrak{g}$ , we find

$$\{\mathcal{K}_{0\underline{1}}, \mathcal{K}_{0\underline{2}}\} = -\frac{1}{K}[C_{\underline{12}}, \mathcal{K}_{0\underline{2}}]\delta_{\sigma\sigma'} - \frac{2k}{K}C_{\underline{12}}\partial_\sigma\delta_{\sigma\sigma'}, \tag{3.9a}$$

$$\{\mathcal{K}_{0\underline{1}}, \mathcal{K}_{1\underline{2}}\} = -\frac{1}{K}[C_{\underline{12}}, \mathcal{K}_{1\underline{2}}]\delta_{\sigma\sigma'} + \frac{1+k^2+A^2}{K}C_{\underline{12}}\partial_\sigma\delta_{\sigma\sigma'}, \tag{3.9b}$$

$$\{\mathcal{K}_{1\underline{1}}, \mathcal{K}_{1\underline{2}}\} = \frac{k^2+A^2}{K}[C_{\underline{12}}, \mathcal{K}_{0\underline{2}}]\delta_{\sigma\sigma'} + \frac{2k}{K}[C_{\underline{12}}, \mathcal{K}_{1\underline{2}}]\delta_{\sigma\sigma'} - \frac{2k}{K}C_{\underline{12}}\partial_\sigma\delta_{\sigma\sigma'}. \tag{3.9c}$$

In the  $\mathfrak{su}(2)$  case, this Poisson bracket is exactly the one of the ‘squashed WZW model’ which was identified in [13], up to the overall factor of  $1/K$ . The bracket (3.9) was also studied more recently in [9] for a general Lie algebra. The notation used there for the current components  $\mathcal{I}_0, \mathcal{I}_1$ , and for the parameters  $\rho, x$  and  $e^2$  may be identified with the present notation as

$$\begin{aligned} \mathcal{K}_0 &= -\mathcal{I}_0, & \mathcal{K}_1 &= -\mathcal{I}_1, & K &= \frac{1}{2e^2} \frac{1}{1 + \rho^2 + x(1 - \rho^2)}, \\ k &= 2e^2 K\rho, & A^2 &= 4e^4 K^2(1 - x^2)(1 - \rho^2)^2. \end{aligned}$$

In particular, since  $A$  must be a real parameter in the action (2.1), we have  $A^2 \geq 0$  and so we see that the Poisson brackets (3.9) correspond to  $-1 \leq x \leq 1$ , which is known to correspond to the complex branch [2].

Let us consider the  $k = 0$  limit of these Poisson brackets. If we choose the normalisation of the action to be  $K = 1 + \eta^2$  then we find

$$\{\mathcal{K}_{0\underline{1}}, \mathcal{K}_{0\underline{2}}\} = -\frac{1}{1 + \eta^2}[C_{\underline{12}}, \mathcal{K}_{0\underline{2}}]\delta_{\sigma\sigma'}, \tag{3.10a}$$

$$\{\mathcal{K}_{0\underline{1}}, \mathcal{K}_{1\underline{2}}\} = -\frac{1}{1 + \eta^2}[C_{\underline{12}}, \mathcal{K}_{1\underline{2}}]\delta_{\sigma\sigma'} + C_{\underline{12}}\partial_\sigma\delta_{\sigma\sigma'}, \tag{3.10b}$$

$$\{\mathcal{K}_{1\underline{1}}, \mathcal{K}_{1\underline{2}}\} = \frac{\eta^2}{1 + \eta^2}[C_{\underline{12}}, \mathcal{K}_{0\underline{2}}]\delta_{\sigma\sigma'}. \tag{3.10c}$$



This agrees with the one-parameter deformation of the Poisson brackets of the principal chiral model (see for instance [3]).

### 3.4. Twist function

The Poisson brackets (3.9) can be written in terms of the Lax matrix (2.20) as

$$\{\mathcal{L}_1(z), \mathcal{L}_2(z')\} = [r_{12}, \mathcal{L}_1(z) + \mathcal{L}_2(z')] \delta_{\sigma, \sigma'} - [s_{12}, \mathcal{L}_1(z) - \mathcal{L}_2(z')] \delta_{\sigma, \sigma'} - 2s_{12} \delta'_{\sigma \sigma'}, \tag{3.11}$$

where the  $r/s$ -matrices read

$$r_{12}(z, z') = \frac{\varphi_{\eta, k}(z)^{-1} + \varphi_{\eta, k}(z')^{-1}}{z - z'} C_{12}, \quad s_{12}(z, z') = \frac{\varphi_{\eta, k}(z)^{-1} - \varphi_{\eta, k}(z')^{-1}}{z - z'} C_{12},$$

and the deformed twist function  $\varphi_{\eta, k}(z)$  is given by

$$\varphi_{\eta, k}(z) = \frac{K(1 - z^2)}{A^2 + (z - k)^2}. \tag{3.12}$$

Here  $A$  is defined as in (2.15). As usual, the form (3.11) for the Poisson bracket of the Lax matrix implies the existence of an infinite number of Poisson commuting quantities [14,15]. Finally, let us note that, just as in the case of the 1-parameter deformed Poisson brackets [1], one can also recast the Poisson bracket (3.9) in the form of two Poisson commuting complex Kac–Moody algebras. In general, Kac–Moody currents can be constructed by taking the Lax matrix at the simple poles of the twist function [18]. In the present case, if we define the currents

$$\mathcal{J}_{\pm} = \mathcal{L}(k \pm iA) = \frac{1}{1 - (k \pm iA)^2} (\mathcal{K}_1 + (k \pm iA)\mathcal{K}_0),$$

then  $\mathcal{J}_-^\dagger = \mathcal{J}_+$  and these satisfy the following Poisson brackets

$$\begin{aligned} \gamma_{\pm} \{\mathcal{J}_{\pm 1}, \mathcal{J}_{\pm 2}\} &= -[C_{12}, \mathcal{J}_{\pm 2}] \delta_{\sigma \sigma'} + C_{12} \delta'_{\sigma \sigma'}, \\ \{\mathcal{J}_{+1}, \mathcal{J}_{-2}\} &= 0, \end{aligned}$$

where  $\gamma_{\pm} = \pm \frac{K}{2iA} (1 - (k \pm iA)^2)$ .

## 4. Conclusion

In this note we presented a two-parameter deformation of the principal chiral model.

Let us emphasise that our proof of the integrability of the model defined by the action (2.1) makes use of the identity (2.5) for the  $R$ -matrix. Although the latter is certainly satisfied by the standard  $R$ -matrix, it doesn't hold in general. The more general case deserves further study.

The deformed model furnishes a higher rank generalisation of the ‘squashed WZNW-model’ introduced in [12,13]. The action of the latter is given simply by adding a Wess–Zumino term to the squashed 3-sphere  $\sigma$ -model action. A peculiarity of the  $\mathfrak{su}(2)$  case, however, is the absence of a  $B$ -field in the Yang–Baxter  $\sigma$ -model. In the higher rank cases, where a  $B$ -field is present, we see that the action (2.1) is not obtained from the Yang–Baxter  $\sigma$ -model action by the mere addition of a Wess–Zumino term. Indeed, the relative weight of the metric and  $B$ -field terms in the Yang–Baxter  $\sigma$ -model action also has to be suitably deformed.

As already emphasised, the model provides a realisation of the two-parameter family of deformed Poisson brackets [11] in the complex branch. It would be very interesting to identify the integrable  $\sigma$ -model realising the deformed Poisson bracket in the real branch. Although the latter is not known, it was recently shown in [9] that this hypothetical model admits a classical Yangian symmetry  $Y_C(\mathfrak{g})$ . It would be interesting to identify the full symmetry algebra of the model in the complex branch as well, using the methods developed in [6,7,3,12,19].

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## References

- [1] S. Rajeev, Nonabelian bosonization without Wess–Zumino terms. I. New current algebra, *Phys. Lett. B* 217 (1989) 123.
- [2] T.J. Hollowood, J.L. Miramontes, D.M. Schmidt, Integrable deformations of strings on symmetric spaces, arXiv:1407.2840.
- [3] F. Delduc, M. Magro, B. Vicedo, On classical  $q$ -deformations of integrable  $\sigma$ -models, *J. High Energy Phys.* 1311 (2013) 192, arXiv:1308.3581.
- [4] C. Klimcik, Yang–Baxter sigma models and dS/AdS T duality, *J. High Energy Phys.* 0212 (2002) 051, arXiv:hep-th/0210095.
- [5] C. Klimcik, On integrability of the Yang–Baxter  $\sigma$ -model, *J. Math. Phys.* 50 (2009) 043508, arXiv:0802.3518.
- [6] I. Kawaguchi, K. Yoshida, Hybrid classical integrability in squashed sigma models, *Phys. Lett. B* 705 (2011) 251–254, arXiv:1107.3662.
- [7] I. Kawaguchi, T. Matsumoto, K. Yoshida, On the classical equivalence of monodromy matrices in squashed sigma model, *J. High Energy Phys.* 1206 (2012) 082, arXiv:1203.3400.
- [8] K. Sfetsos, Integrable interpolations: From exact CFTs to non-Abelian T-duals, *Nucl. Phys. B* 880 (2014) 225–246, arXiv:1312.4560.
- [9] G. Itsios, K. Sfetsos, K. Siampos, A. Torrielli, The classical Yang–Baxter equation and the associated Yangian symmetry of gauged WZW-type theories, *Nucl. Phys. B* 889 (2014) 64–86, arXiv:1409.0554.
- [10] K. Sfetsos, D.C. Thompson, Spacetimes for  $\lambda$ -deformations, arXiv:1410.1886.
- [11] J. Balog, P. Forgacs, Z. Horvath, L. Palla, A new family of SU(2) symmetric integrable sigma models, *Phys. Lett. B* 324 (1994) 403–408, arXiv:hep-th/9307030.
- [12] I. Kawaguchi, D. Orlando, K. Yoshida, Yangian symmetry in deformed WZNW models on squashed spheres, *Phys. Lett. B* 701 (2011) 475–480, arXiv:1104.0738.
- [13] I. Kawaguchi, K. Yoshida, A deformation of quantum affine algebra in squashed Wess–Zumino–Novikov–Witten models, *J. Math. Phys.* 55 (2014) 062302, arXiv:1311.4696.
- [14] J.M. Maillet, Kac–Moody algebra and extended Yang–Baxter relations in the  $O(N)$  non-linear sigma model, *Phys. Lett. B* 162 (1985) 137.
- [15] J.M. Maillet, New integrable canonical structures in two-dimensional models, *Nucl. Phys. B* 269 (1986) 54.
- [16] B. Vicedo, The classical R-matrix of AdS/CFT and its Lie dialgebra structure, *Lett. Math. Phys.* 95 (2011) 249–274, arXiv:1003.1192.
- [17] C. Klimcik, Integrability of the bi-Yang–Baxter  $\sigma$ -model, *Lett. Math. Phys.* 104 (2014) 1095–1106, arXiv:1402.2105.
- [18] F. Delduc, M. Magro, B. Vicedo, in preparation.
- [19] I. Kawaguchi, T. Matsumoto, K. Yoshida, The classical origin of quantum affine algebra in squashed sigma models, *J. High Energy Phys.* 1204 (2012) 115, arXiv:1201.3058.